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KK spectroscopy of type IIB supergravity on $AdS_5 \times T^{11}$

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Abstract: We give full details for the computation of the Kaluza-Klein mass spectrum of type IIB Supergravity on $AdS_5 \times T^{11}$, with $T^{11} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1)$, that has recently lead to both stringent tests and interesting predictions on the AdS_5/CFT_4 correspondence for $\mathcal{N} = 1$ SCFT's [1]. We exhaustively explain how KK states arrange into $\mathrm{SU}(2,2|1)$ supermultiplets, and stress some relevant features of the T^{11} manifold, such as the presence of topological modes in the spectrum originating from the existence of non-trivial 3-cycles. The corresponding Betti vector multiplet is responsible for the extra baryonic symmetry in the boundary CFT. More generally, we use the simple T^{11} coset as a laboratory to revive the technique and show the power of KK harmonic expansion, in view of the present attempts to probe along the same lines also M-theory compactifications and the AdS_4/CFT_3 map.

Keywords: Field Theories in Higher Dimensions, Superstrings and Heterotic Strings, Space-Time Symmetries, Supergravity Models.

Contents

1. Introduction

Knowledge of the full Kaluza-Klein (KK) mass spectrum of type IIB Supergravity compactified on the product of AdS_5 spacetime and the coset manifold $T^{11} = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{U}(1)}$ has recently been crucially used to perform accurate spectroscopic tests as well as obtain many new predictions [1] on the AdS/CFT correspondence [2] by comparison with a specific $\mathcal{N} = 1$, four-dimensional SCFT [3].

Although partial results concerning the T^{11} laplacean [4], the first-order operators acting on fermions and on the 2-form [5] were already quite inspiring, only from studying every sector of the spectrum one can fully analyse the multiplet structure and shortening patterns that allow to establish a precise mapping between KK states and conformal operators in the boundary field theory.

While the focus in [1] was on the AdS/CFT correspondence, this paper is totally devoted to the techniques involved in performing harmonic analysis on T^{11}, that were just touched in that previous work.

Harmonic analysis, applied to the KK reduction of higher supergravity theories, has been thoroughly developed in the past (see [6, 7, 8, 9] and references therein). Nowadays, it seems to live a new youth, especially in connection with the basic rôle played by all AdS compactified supergravity, and particularly of M-theory.

The T^{11} manifold provides a simple and effective example of coset manifold with Killing spinors that can be used to show the general theory of computing KK spectra at work. Thus we briefly expose it here in the essential steps without any claim for mathematical rigour. This can perhaps make more evident points presented in [1] and might be of some help to those who are not aware of the method.

Since we determine the full KK spectrum, we organize it in $\mathrm{SU}(2,2|1)$ supermultiplets and discuss some interesting features arising in this kind of analysis, we think that this paper could provide a self-contained reference for analogous computations in different examples of KK harmonic expansion that are nowadays under investigation, such as M-theory on $AdS_4 \times M_7$ [10], with M_7 being either M^{111} spaces [11, 12, 13, 14], the real Stiefel manifold $V_{(5,2)}$ [15] or the N^{010} and Q^{111} spaces [16].

The strength of our analysis lies in the fact that, in view of the AdS/CFT correspondence, our results provide an accurate check of the field theory independently derived by orbifold resolution techniques [3, 17, 18]. This is not the case for the AdS_4/CFT_3 correspondence, where the supergravity side can be deeply investigated and is partly well known, while the relevant three-dimensional conformal field theories are not still understood, although there is some work in this direction [19, 20, 21]. The KK analisys acquires than a prominent position, since it can give us the right hints for building such dual CFT's.

Beside the harmonic analysis and the KK mass spectrum, we also give some emphasis to the existence on T^{11} of a "Betti" vector multiplet [13, 23], that is a gauge vector multiplet which is a singlet of the full isometry group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}_R(1)$ and whose presence is related to the non-trivial Betti numbers $b_2 = b_3 = 1$ on T^{11}. As already observed in [1, 25], the existence of such kind of multiplets is related to the possibility of wrapping a p-brane ($p = 3$ in our case) on the non-trivial p-cycles of the internal manifold.

A new feature of our five-dimensional theory is that there appears not only the single vector field associated to the non-trivial 3-form, but, as we will see, there is also a tensor and a scalar field, which are part of a Betti tensor and a Betti hypermultiplet respectively.

In section 2 we review in some more detail with respect to previous papers the geometry of T^{pq} and specifically of T^{11} (which turns out to be the only supersymmetric one in the family where the integers p and q define the possible embeddings of the $\mathrm{U}(1)$ in the denominator), establishing conventions and notations. In section 3 we give our essential resumé of harmonic expansion on a general coset manifold, giving at each step of the construction, the application to our specific manifold. The KK procedure for T^{11} is then worked out and the mass spectrum thoroughly computed in section 4. In section 5 we explain how to reconstruct the supermultiplets of $SU(2,2|1)$ from the eigenvalues of the invariant differential operators and from the group theoretical knowledge of the $SU(2,2|1)$ representations. The results are collected in tables 2–10. In section 6 we recall the main properties of the Betti multiplets as they were first introduced in [23] and we discuss their explicit form in our specific case.

We also present two appendices. In the appendix A notations and conventions are collected, while in appendix B we show how to compute the differential operators on a coset manifold in a purely algebraic way, which is the heart of the general method for computing mass spectra in KK theories.

2. T^{11} geometry

The T^{pq} spaces [26] are the coset manifolds

$$\frac{G}{H} = \frac{SU(2) \times SU(2)}{U_H(1)},$$

where the $U_H(1)$ generator T_H is embedded in the two $SU(2)$ factors as the linear combination

$$T_H \equiv p\, T_3 + q\, \hat{T}_3\,, \qquad p, q \in \mathbb{N} \text{ coprime} \tag{2.1}$$

where T_3 and $\hat{T}_3$ generate $U(1)$ subgroups of the two $SU(2)$ in G.

To describe the geometry of these varieties, we can take two copies of the $SU(2)$ algebra with generators[1] T_A, $\hat{T}_A$, $(A = 1, \ldots, 3)$:

$$[T_A, T_B] = \epsilon_{AB}{}^C T_C\,, \qquad [\hat{T}_A, \hat{T}_B] = \epsilon_{AB}{}^C \hat{T}_C\,, \tag{2.2}$$

and write the Maurer-Cartan equations (MCe's) for their dual forms e^A, $\hat{e}^A$:

$$\begin{aligned} de^A + \frac{1}{2}\epsilon_{BC}{}^A\, e^B e^C &= 0\,, \\ d\hat{e}^A + \frac{1}{2}\epsilon_{BC}{}^A\, \hat{e}^B \hat{e}^C &= 0\,. \end{aligned} \tag{2.3}$$

Choosing the two $U(1)$ subgroups of $SU(2)$ to be generated by e^3 and $\hat{e}^3$, we may rewrite $(A = (i, 3)$, $\hat{A} = (r, 3)$,$i, r = 1, 2)$

$$de^i + \epsilon^i{}_j\, e^3 e^j = 0\,, \tag{2.4a}$$

$$d\hat{e}^r + \epsilon^r{}_s\, \hat{e}^3 \hat{e}^s = 0\,, \tag{2.4b}$$

$$de^3 + \frac{1}{2}\epsilon_{ij}\, e^i e^j = 0\,, \tag{2.4c}$$

$$d\hat{e}^3 + \frac{1}{2}\epsilon_{rs}\, \hat{e}^r \hat{e}^s = 0\,. \tag{2.4d}$$

[1] The conventions used for the ϵ symbol, the group metrics etc. are reported in appendix A.

Passing to the quotient $G \to G/H$, the MCe's for the left-invariant one-forms on G become the corresponding MCe's for left-invariant forms on the coset G/H. The linear combination

$$\omega \equiv \frac{p\, e^3 + q\, \hat{e}^3}{p^2+q^2} \tag{2.5}$$

dual to T_H defined in (2.1) can be identified as the H-connection of G/H, while e^i, $\hat{e}^r$ and the orthogonal combination

$$e^5 = \frac{p\, e^3 - q\, \hat{e}^3}{p^2+q^2} \tag{2.6}$$

can be identified with the five vielbeins spanning the cotangent space to G/H.

We do not dwell with general values of p and q because we are interested only in T^{pq} spaces endowed with supersymmetry and it has been shown in [26] that this happens only if $p = q = 1$. Thus we take

$$T_H \equiv T_3 + \hat{T}_3\,, \qquad \omega \equiv \frac{e^3 + \hat{e}^3}{2}\,, \qquad e^5 \equiv \frac{e^3 - \hat{e}^3}{2}\,, \tag{2.7}$$

and

$$T_5 = T_3 - \hat{T}_3\,, \tag{2.8}$$

where T_5 is the coset generator dual to e^5. In the (ω, e^5) basis, the MCe's (2.4c)–(2.4d) become

$$d\omega + \frac{1}{4}\epsilon_{ij} e^i e^j - \frac{1}{4}\epsilon_{rs}\hat{e}^r\hat{e}^s = 0\,, \tag{2.9}$$

$$de^5 + \frac{1}{4}\epsilon_{ij} e^i e^j + \frac{1}{4}\epsilon_{rs}\hat{e}^r\hat{e}^s = 0\,. \tag{2.10}$$

It is convenient to introduce rescaled vielbeins $V^a \equiv (V^i, V^s, V^5)$:

$$V^i = a\, e^i\,, \qquad V^s = b\, \hat{e}^s\,, \qquad V^5 = c\, e^5\,, \tag{2.11}$$

where a, b, c are real rescaling factors which will be determined by requiring that T^{11} be an Einstein space [8, 27]. Using (2.11) in (2.4a)–(2.4b) and (2.9)–(2.10), we get the full set of MCe's of T^{11} for the vielbein V^a and the H-connection ω

$$\begin{aligned} dV^i + \epsilon^{ij}(\omega + cV^5)V_j &= 0\,, \\ dV^r + \epsilon^{rs}(\omega - cV^5)V_s &= 0\,, \\ dV^5 + \frac{a^2}{4c}\epsilon^{ij}V_iV_j - \frac{b^2}{4c}\epsilon^{rs}V_rV_s &= 0\,, \\ d\omega + \frac{a^2}{4}\epsilon^{ij}V_iV_j + \frac{b^2}{4}\epsilon^{rs}V_rV_s &= 0\,. \end{aligned} \tag{2.12}$$

Once we have the vielbeins, we may construct the Riemann connection one-form $\mathcal{B}^{ab} \equiv -\mathcal{B}^{ba}$ $(a, b = i, s, 5)$, imposing the torsion-free condition

$$dV^a - \mathcal{B}^{ab}V_b = 0\,. \tag{2.13}$$

JHEP11(1999)009

It is straightforward to compare the equations one obtains from (2.13) with the rescaled MCe's, finding

$$
\begin{aligned}
\mathcal{B}^{ij} &= -\epsilon^{ij}\left[\omega + \left(c - \frac{a^2}{4c}\right)V^5\right], & \mathcal{B}^{5i} &= \frac{a^2}{4c}\,\epsilon^{ij}\,V_j\,,\\
\mathcal{B}^{st} &= -\epsilon^{st}\left[\omega - \left(c - \frac{b^2}{4c}\right)V^5\right], & \mathcal{B}^{5s} &= -\frac{b^2}{4c}\,\epsilon^{st}\,V_t\,,\\
\mathcal{B}^{is} &= 0\,. && \qquad (2.14)
\end{aligned}
$$

The Riemann tensor obtained from the definition of the curvature 2-form:

$$
R^{ab} \equiv d\mathcal{B}^{ab} - \mathcal{B}^{a}{}_{c}\mathcal{B}^{cb} = R^{ab}{}_{cd}V^cV^d\,, \qquad (2.15)
$$

has components given by

$$
\begin{aligned}
R^i{}_{5j5} &= -\delta^i_j\frac{a^2b^2}{32c^2}\,,\\
R^i{}_{sjt} &= \frac{1}{8}\left(\frac{a^2b^2}{4c^2}\right)\epsilon^i{}_j\epsilon_{st}\,,\\
R^i{}_{jkl} &= \epsilon^i{}_j\epsilon_{kl}\left(\frac{a^2}{2} - \frac{a^2b^2}{16c^2}\right) + \frac{a^2b^2}{16c^2}\epsilon^i{}_{[k}\epsilon_{l]j}\,,\\
R^s{}_{5t5} &= -\delta^s_t\frac{a^2b^2}{32c^2}\,,\\
R^s{}_{trz} &= \epsilon^s{}_t\epsilon_{rz}\left(\frac{a^2}{2} - \frac{a^2b^2}{16c^2}\right) + \frac{a^2b^2}{16c^2}\epsilon^s{}_{[r}\epsilon_{z]t}\,,\\
R^i{}_{jst} &= \frac{a^2b^2}{16c^2}\epsilon^i{}_j\epsilon_{st}\,. \qquad (2.16)
\end{aligned}
$$

The requirement for T^{11} to be an Einstein space means that its Ricci tensor must be

$$
R^a{}_b = 2\,e^2\,\delta^a_b\,, \qquad (2.17)
$$

where e is the constant VEV of the IIB four-form self-dual field strength, needed to obtain a spontaneous KK compactification via the Freund-Rubin mechanism [28]. It is well known that e^2 is identified with the cosmological constant of AdS_5 (see (3.47) below). The Ricci tensor components derived from (2.16) are given by

$$
R^i{}_k = \left(\frac{1}{2}a^2 - \frac{a^4}{16c^2}\right)\delta^i_k\,, \qquad R^s{}_t = \left(\frac{1}{2}b^2 - \frac{b^4}{16c^2}\right)\delta^s_t\,, \qquad R^5{}_5 = \frac{a^4}{8c^2}\,, \qquad (2.18)
$$

and in order to satisfy (2.17) we must have

$$
a^2 = b^2 = 6e^2\,, \qquad \text{and} \qquad c^2 = \frac{9}{4}e^2\,. \qquad (2.19)
$$

This fixes the square of the rescalings and thus their absolute value, but not their sign. We will see that this is going to be determined by supersymmetry requirement.

Before proceeding, we would like to briefly mention an alternative approach for describing the geometry of the T^{11} manifold developed in [27] and used for example in [4]. If we decompose the SU(2) × SU(2) Lie algebra $\mathbb{G}$ with respect to the T_H generator as $\mathbb{G} = \mathbb{H} + \mathbb{K}$, where $\mathbb{H}$ is made of the single T_H generator and the coset algebra $\mathbb{K}$ contains the T_i, $\hat{T}_s$ and T_5 generators, the commutation relations between the $\mathbb{G}$ generators are

$$\begin{aligned} [T_i, T_j] &= \frac{1}{2}\epsilon_{ij}(T_H + T_5)\,, \qquad [\hat{T}_s, \hat{T}_t] = \frac{1}{2}\epsilon_{st}(T_H - T_5)\,, \\ [T_5, T_i] &= [T_H, T_i] = \epsilon_i{}^j T_j\,, \qquad [T_5, \hat{T}_s] = [T_H, \hat{T}_s] = \epsilon_s{}^t \hat{T}_t\,, \\ [T_i, \hat{T}_s] &= [T_5, T_H] = 0\,. \end{aligned} \tag{2.20}$$

From these, one can derive the structure constants

$$\begin{aligned} C_{ij}{}^H &= C_{ij}{}^5 = \frac{1}{2}\epsilon_{ij}\,, \qquad C_{st}{}^H = C_{st}{}^5 = \frac{1}{2}\epsilon_{st}\,, \\ C_{5i}{}^j &= C_{Hi}{}^j = \epsilon_i{}^j\,, \qquad C_{5s}{}^t = C_{Hs}{}^t = \epsilon_s{}^t\,, \\ C_{is}{}^a &= C_{5H}{}^a = 0\,. \end{aligned} \tag{2.21}$$

and find again the Riemann tensor components (2.16) from the formula [27]

$$\begin{aligned} R^a{}_{b,de} = {}& \frac{1}{4}C^a_{bc}C^c_{de}\begin{pmatrix} a\ b \\ c \end{pmatrix}\frac{r(d)r(e)}{r(c)} + \frac{1}{2}C^a_{bH}C^H_{de}r(d)r(e) + \\ & + \frac{1}{8}C^a_{cd}C^c_{be}\begin{pmatrix} a\ c \\ d \end{pmatrix}\begin{pmatrix} b\ c \\ e \end{pmatrix} - \frac{1}{8}C^a_{ce}C^c_{bd}\begin{pmatrix} a\ c \\ e \end{pmatrix}\begin{pmatrix} b\ c \\ d \end{pmatrix}, \end{aligned} \tag{2.22}$$

where

$$\begin{pmatrix} a\ b \\ c \end{pmatrix} \equiv \frac{r(a)r(c)}{r(b)} + \frac{r(b)r(c)}{r(a)} - \frac{r(a)r(b)}{r(c)}\,, \tag{2.23}$$

and $r(a) = \{a, b, c\}$.

3. Harmonic expansion

3.1 The general theory

In this chapter we give a resumé of the general theory of harmonic expansion on coset manifolds (see e.g. [6, 7, 8, 9, 14] and references therein) and adapt the general formulae to our specific case, namely compactification of type IIB supergravity on the T^{11} manifold.

In Kaluza-Klein theories, we are faced with the problem of computing the mass spectrum of a D-dimensional model compactified down to $D-d$ dimensions, d being the dimensions of a compact space, usually of the form of a coset G/H.

We take the space-time to be AdS_{D-d}, i.e. anti-de Sitter space in $D-d$ dimensions.

The first thing to do is to compute the fluctuations of the D-dimensional fields around a particular background [12, 23] which, in spontaneous compactifications, turns out to be a solution of the D-dimensional theory equations of motion. After the linearisation of the D-dimensional equations of motion has been performed, we are left, for each field of the theory, with an equation of the type:

$$\left(\Box_x^{\{J\}} + \boxtimes_y^{[\lambda]}\right) \Phi^{\{J\}[\lambda]}(x,y) = 0\,. \tag{3.1}$$

where $\Box$ and $\boxtimes$ are the kinetic operators in the AdS space-time (x-coordinates) and in the compact space (y-coordinates). The label $\{J\}$ denotes the AdS_{D-d} quantum numbers of the field (scalar, vector, spinor, etc.), while the SO(d) representation in the tangent space to G/H is labeled by the corresponding Young diagram[2] $[\lambda] \equiv [\lambda_1, \ldots, \lambda_{[d/2]}]$.

The important thing about (3.1) is that the differential operator symbolically denoted by $\boxtimes_y^{[\lambda]}$ is a Laplace-Beltrami operator on G/H. This means that, beside being invariant under G, its eigenfunctions, the harmonics on G/H, define a complete set of functions for the fields $\Phi^{[\lambda]}(y)$ on G/H and, for each eigenvalue, they span an irreducible representation of G.

The invariant operators $\boxtimes$ can all be constructed in terms of the (flat) covariant derivatives $\mathcal{D}_a$ ($a = 1, \ldots d$) on G/H and the invariant G-tensors. Let us give a list of these operators for various irrepses $\left[\lambda_1, \ldots, \lambda_{[d/2]}\right]$ appearing in the KK compactification[3]

$$\begin{aligned}
\boxtimes_y Y_{[0,0,\ldots,0]} &\equiv \Box Y\,, && (3.2\text{a})\\
\boxtimes_y Y_{[1,0,\ldots,0]} &\equiv 2\mathcal{D}^a \mathcal{D}_{[a} Y_{b]}\,, && (3.2\text{b})\\
\boxtimes_y Y_{[1,\ldots,1,0,\ldots 0]} &\equiv (p+1)\mathcal{D}^{a_{p+1}} \mathcal{D}_{[a_1} Y_{a_2 \ldots a_{p+1}]}\,, && (3.2\text{c})\\
\boxtimes_y Y_{[2,0,\ldots,0]} &\equiv 3\mathcal{D}^c \mathcal{D}_{(c} Y_{ab)}\,, && (3.2\text{d})\\
\boxtimes_y Y_{[1/2,1/2,\ldots,1/2]} &\equiv \not{\mathcal{D}}\,\Xi\,, && (3.2\text{e})\\
\boxtimes_y Y_{[3/2,1/2,\ldots,1/2]} &\equiv \gamma^{abc}\, \mathcal{D}_b\, \Xi_c\,. && (3.2\text{f})
\end{aligned}$$

Note that all the Laplace-Beltrami operators are either second-order differential operators acting on the bosonic harmonics Y (which can be expressed in terms of the covariant laplacean $\Box$ on G/H plus curvature terms) or they are the first-order operators acting on spinor harmonics Ξ (in this case they can be expressed in terms of the Dirac operator $\not{\mathcal{D}}$ on G/H).

We point out that for particular values of the internal space dimension $d = 2k+1$, the Laplace-Beltrami operator acting on a k-form can be written in terms of the first-order operator $\star d$. For example, in $d = 5$, the Laplace-Beltrami operator on the

[2] We will mostly omit the $\{J\}$ and $[\lambda]$ labels when not necessary.

[3] Symmetrisation and antisymmetrisation are understood with weight one; we do not write explicitly the spinor index on the spinor harmonics: $\Xi \equiv \Xi^\alpha$.

2-forms $Y_{ab}V^aV^b$ is

$$\boxtimes_y Y_{[1,1]} \equiv \star dY_{ab}V^aV^b = \frac{1}{2}\epsilon_{abcde}D^cY^{de}\,V^aV^b\,, \tag{3.3}$$

and the usual second-order operator is simply the square of the first-order one.

To define the harmonics we start from the fundamental equation for the coset representative $L(y)$ of G/H:

$$gL(y) = L(y')h(y,g)\,, \tag{3.4}$$

where $g \in G$, $h \in H$ and y, y' are two points of G/H related by the g transformation. In a given G-representation $\mathbb{D}$ of indices m, n we can rewrite (3.4) as follows

$$\mathbb{D}(g)^m{}_n\mathbb{D}^n(L(y))_{k_i} = \mathbb{D}^m(L(y'))_{h_i}\mathbb{D}(h(y,g))^{h_i}{}_{k_i}\,, \tag{3.5}$$

where the N-dimensional range of the indices m, n of the representation space of $\mathbb{D}(h)$ has been fragmented into subsets h_i corresponding to their i-th irreducible fragment; in other words, if $\{\nu\}$ identifies the G-representation and $\{\alpha_i\}$ identifies one of the irreducible H-representations according to the branching rule

$$\{\nu\} \to \{\alpha_1\} \oplus \{\alpha_2\} \oplus \cdots \oplus \{\alpha_M\}\,, \tag{3.6}$$

then we have

$$m = \{h_1, \ldots, h_M\}\,, \qquad h_i = 1, \ldots, n_i\,, \qquad \sum_1^M h_i = \dim \mathbb{D}. \tag{3.7}$$

We now define as irreducible harmonics

$$\left[Y^{\{\nu\}}_{\{\alpha_i\}}(y)\right]^m{}_{h_i} \equiv \mathbb{D}^m{}_{h_i}(L^{-1}(y))\,. \tag{3.8}$$

The functions $Y^{(\nu)m}_{\{\alpha_i\}h_i}$, for fixed $\{\alpha_i\}$, are a complete set of functions for the expansion of a field $\Phi_{h_i}(y)$ on G/H, where $\Phi_{h_i}(y)$ transforms in the irrep $\{\alpha_i\}$ of H. However, in KK, the D-dimensional physical fields also depend on the space-time coordinates x, so that a generic field $\Phi_{h_i}(x,y)$, transforming in the irrep $\{\alpha_i\}$, can be expanded as

$$\Phi_{h_i}(x,y) = \sum_{\{\nu\}}\sum_m \Phi^{\{\nu\}}_{mh_i}(x)(Y^{\{\nu\}}(y))^m_{h_i}\,. \tag{3.9}$$

where for notational simplicity we have suppressed the index α_i referring to the particular H-representation.

Once the G-representation $\{\nu\}$ is fixed we can still have a state degeneration. Indeed there are cases in which the same G-representation, with the same H quantum numbers can be obtained in many ways. In such cases the correct field expansion (3.9) is replaced by

$$\Phi_{h_i}(x,y) = \sum_{\{\nu\}}\sum_m\sum_\delta \Phi^{\{\nu\}\delta}_{mh_i}(x)(Y^{\{\nu\}\delta}(y))^m_{h_i}\,, \tag{3.10}$$

where δ counts the state degeneracy.

We now exemplify the previous discussion with our specific case, where the coset is $G/H = (\mathrm{SU}(2)\times\mathrm{SU}(2))/\,\mathrm{U}_H(1)$ and $\dim(G/H) = 5$. Thus $\{\nu\}$ is identified with (j,l), the quantum numbers of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ irreducible representations, and $\{\alpha_i\}$ is identified with the charge q_i of the i-th one-dimensional fragment of the branching of a given $\mathrm{SU}(2)\times\mathrm{SU}(2)$ representation under $\mathrm{U}_H(1)$.

To be more explicit, we write a generic representation of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ in the Young tableaux formalism:

$$(j,l) \equiv \underbrace{\boxed{}\boxed{\cdots}\boxed{}}_{2j} \otimes \underbrace{\boxed{}\boxed{\cdots}\boxed{}}_{2l}\,. \tag{3.11}$$

A particular component of (3.11) can be written as

$$\underbrace{\boxed{1}\boxed{\cdots}\boxed{1}}_{m_1}\underbrace{\boxed{2}\boxed{\cdots}\boxed{2}}_{m_2} \otimes \underbrace{\boxed{1}\boxed{\cdots}\boxed{1}}_{n_1}\underbrace{\boxed{2}\boxed{\cdots}\boxed{2}}_{n_2} \tag{3.12}$$

and we have

$$\begin{cases} 2j = m_1 + m_2\,, \\ 2j_3 = m_2 - m_1\,, \end{cases} \qquad \begin{cases} 2l = n_1 + n_2\,, \\ 2l_3 = n_2 - n_1\,. \end{cases} \tag{3.13}$$

Furthermore (recalling the definitions (2.1)–(2.8)) we get

$$\begin{aligned} T_H Y^{(j,l,r)}_{(q)} &= \mathrm{i}\,q\,Y^{(j,l,r)}_{(q)} \equiv \mathrm{i}\,(j_3 + l_3)\,Y^{(j,l,r)}_{(q)}\,, \\ T_5 Y^{(j,l,r)}_{(q)} &= \mathrm{i}\,r\,Y^{(j,l,r)}_{(q)} \equiv \mathrm{i}\,(j_3 - l_3)\,Y^{(j,l,r)}_{(q)}\,. \end{aligned} \tag{3.14}$$

Hence

$$\begin{aligned} 2j_3 &= q + r \equiv m_2 - m_1\,, \\ 2l_3 &= q - r \equiv n_2 - n_1\,. \end{aligned} \tag{3.15}$$

It is easy to see that, given a generic $\{\nu\} = (j,l)$ G-representation and fixed the H quantum number q, we have still the freedom to chose how to place the 1 and 2's in the boxes of the Young tableaux. We have therefore a state degeneration. To remove such a degeneration we classify the various states by means of a fourth quantum number r which is the charge of the state under the normalizer $\mathrm{U}(1)$ in H. This $\mathrm{U}(1)$ coincides with the R-symmetry factor in the full isometry group $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. We point out that this number is not a good quantum number for a fixed representation as it is not the same for all the q-fragments appearing in the expansion of a generic (spinor) tensor, but it is useful to determine the eigenvalues of the mass operators on the harmonics in terms of the known quantum numbers of the isometry group.

Since in this case the h_i index is one-dimensional, we can write the generic harmonic of T^{11} as $Y^{(j,l,r)}_{(q_i)}(y)^m$ and the expansion of a field belonging to the subspace of charge (q_i) as

$$\Phi_{(q_i)}(x,y) = \sum_{(j,l)}\sum_{m}\sum_{r} \Phi^{(j,l,r)}_{q_i}(x)_m (Y^{(j,l,r)}(y))^m_{q_i}\,. \tag{3.16}$$

Coming back to the general case, we are now able to constructing the eigenfunctions of the Laplace-Beltrami operators acting on the $\mathrm{SO}(d)$ tangent group tensor and spinor fields. Let us denote by $\Phi^{[\lambda_1,\ldots,\lambda_{[d/2]}]}_{ab\ldots}(x,y)$ such a field where $[\lambda] \equiv [\lambda_1,\ldots,\lambda_{[d/2]}]$ are Young labels of the $\mathrm{SO}(d)$ representation and $ab\ldots$ denote generically some tensor (or spinor) structure of the indices. We note that H is a subgroup of $\mathrm{SO}(d)$ (in our example $H = \mathrm{U}_H(1) \subset \mathrm{SO}(5)$) and so we can branch $[\lambda]$ with respect to H obtaining a set of N irreducible representations of H

$$[\lambda] \xrightarrow{H} \{\beta_1\} + \{\beta_2\} + \cdots \{\beta_N\}\,. \tag{3.17}$$

JHEP11(1999)009

To make this decomposition explicit we observe that if G/H is a d-dimensional coset, then H is a subgroup of $\mathrm{SO}(d)$, its embedding being described by[4]

$$(T_H)^m{}_n = C_H{}^{ab}(T_{ab})^m{}_n\,, \qquad (C_H{}^{ab} = C_{Hc}{}^b\eta^{ca})\,, \tag{3.18}$$

for a given $\mathrm{SO}(d)$ representation labeled by indices m, n. In general, any $\mathrm{SO}(d)$ representation is fully reducible under H:

$$(T_H)^m{}_n = \begin{pmatrix} T_H^{m_1 n_1} & & & \\ & T_H^{m_2 n_2} & & \\ & & \ddots & \\ & & & T_H^{m_M n_M} \end{pmatrix}. \tag{3.19}$$

In particular, the vielbein V^a, which is a d-dimensional $\mathrm{SO}(d)$ vector $([1,0,\ldots,0])$ can be split into fragments transforming irreducibly under H as follows:

$$V^a \to V^{h_1} \oplus \cdots \oplus V^{h_N}\,. \tag{3.20}$$

Turning again to our case $G/H = T^{11}$, the set of generators of the isotropy group $H = \mathrm{U}_H(1)$ is a single generator which is also named T_H. Using the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ algebra given in (2.20) we find the following decomposition of the $\mathrm{SO}(5)$ vector and spinor representation with respect to the one-dimensional subgroup: for the vector representation we find

$$(T_H)_{ab} = C_{Hab} = \left(\begin{array}{c|c|c} \epsilon_{ij} & & \\ \hline & \epsilon_{st} & \\ \hline & & 0 \end{array}\right), \tag{3.21}$$

where each $\epsilon^{ij} = \epsilon^{st} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, while for the spinor representation

$$(T_H) = C_H{}^{ab}(T_{ab}) = -\frac{1}{4}C_H{}^{ab}(\gamma_{ab}) = -\frac{1}{2}(\gamma_{12}+\gamma_{34}) = i\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \tag{3.22}$$

where γ_{ab} are the $\mathrm{SO}(5)$ gamma matrices defined in the appendix A.

Note that the vielbein V^a breaks under $\mathrm{U}_H(1)$ into five one-dimensional fragments $V^i = (V^1, V^2)$, $V^s = (V^3, V^4)$, V^5 with $\mathrm{U}_H(1)$ charges given respectively by $(1,-1); (1,-1); 0$.

Each of the fragments appearing in (3.17) can be expanded in harmonics according to (3.9). Therefore the $\mathrm{SO}(d)$ field $\Phi^{[\lambda]}_{ab\ldots}(x,y)$ (where $ab\ldots$ denote a set of indices labeling the $\mathrm{SO}(d)$ representations according to the Young symbol $[\lambda] = [\lambda_1, \ldots, \lambda_{[d/2]}]$) can be expanded as

$$\Phi^{[\lambda]}_{ab\ldots}(x,y) = \begin{pmatrix} \Phi_{h_1}(x,y) \\ \Phi_{h_2}(x,y) \\ \vdots \\ \Phi_{h_N}(x,y) \end{pmatrix}^{[\lambda]} = \sum_{\{\nu\}}\sum_m\sum_{\xi=1}^N \Phi^{\{\nu\}}(x)_{mh_\xi} \begin{pmatrix} 0 \\ \vdots \\ Y_{h_\xi}(y) \\ \vdots \\ 0 \end{pmatrix}^{\{\nu\},m}, \tag{3.23}$$

[4]To simplify the notations we write $(T)^m{}_n$ instead of the appropriate notation for a representation $(\mathbb{D}(T))^m{}_n$.

where

$$\begin{pmatrix} 0 \\ \vdots \\ Y_{h_\xi} \\ \vdots \\ 0 \end{pmatrix} \tag{3.24}$$

is called a SO(d) harmonic, the most general SO(d) harmonic being

$$Y_{ab\ldots}^{\{\nu\}m}(y) = \begin{pmatrix} Y_{h_1} \\ Y_{h_2} \\ \vdots \\ Y_{h_N} \end{pmatrix}^{\{\nu\}m} . \tag{3.25}$$

It is now evident that in the above expansion the set of irrepses $\{\nu\}$ of G contributing to the expansion are only those that, when branched with respect to H, contain at least one of the irrepses $\{\beta_i\}$ appearing in the decomposition (3.17). In other words, for some i and j we must have $\{\alpha_i\} = \{\beta_j\}$.

It may happen that $\{\nu\}$ contains the same $\{\alpha_\xi\}$ more than once; in this case the index ξ is extended to count also equivalent copies of the same $\{\alpha_\xi\}$ contained in $\{\nu\}$.

At this point we can come back to the (3.1) equation.

Since $\boxtimes_y$ is an invariant Laplace-Beltrami operator on G/H (invariant under the covariant Lie derivatives [8]) we can compute its action on the harmonics $Y_{i_\xi}^{\{\nu\}m}(y)$ obtaining

$$\boxtimes_y Y_{i_\xi}^{\{\nu\}m}(y) = M_{\xi\xi'}^{\{\nu\}} Y_{i_{\xi'}}^{\{\nu\}m}(y) \tag{3.26}$$

so that the linearised equations of motion of the AdS fields become

$$(\delta_{\xi\xi'}\Box_x + M_{\xi\xi'}^{\{\nu\}})\Phi_{n\xi'}^{\{\nu\}}(x) = 0 \tag{3.27}$$

and by diagonalisation of the matrix $M_{\xi\xi'}^{\{\nu\}}$ we find the eigenvalues for the various fragments $\Phi_{n\xi'}^{\{\nu\}}(x)$.

3.2 The mass matrix

Let us now discuss in some detail the computation of the mass matrix $M_{\xi\xi'}^{\{\nu\}}$. We want to show that the action of $\boxtimes_y$ as a differential operator on the harmonics can be reduced to a purely algebraic action in terms of generators of G/H and SO(d).

Since the Laplace-Beltrami operators are constructed in terms of the SO(d) covariant derivatives

$$\mathcal{D} = d + \mathcal{B}^{ab}T_{ab} \equiv d + \mathcal{B}\,, \tag{3.28}$$

where T_{ab} are the SO(d) generators, setting $\mathcal{B} = \omega^H + M$, ($\omega^H$ is defined in the appendix B) one can write

$$\mathcal{D} = \mathcal{D}^H + M\,, \tag{3.29}$$

where the H-covariant derivative is defined by

$$\mathcal{D}^H = d + \omega^H \,. \tag{3.30}$$

The usefulness of the decomposition (3.29)–(3.30), lies in the fact that the action of $\mathcal{D}^H$ on the harmonics can be computed algebraically. Indeed one has quite generally (see appendix B)

$$\mathcal{D}^H = -r(a) T_a V^a \,. \tag{3.31}$$

The covariant derivative, can then be written as

$$\mathcal{D} = (-r(a) V^a T_a + M^{ab} T_{ab}) \,. \tag{3.32}$$

According to the branching of the SO(d) fundamental representation $[1,0,\ldots,0]$ of the vielbein (3.20), we have in any G-representation

$$\mathcal{D}_{h_i} = -r(i) T_{h_i} + M^{ab}_{h_i} T_{ab} \,, \tag{3.33}$$

where T_{ab} are the SO(d) generators (whose normalization in the vector representation is $(T_{ab})^{cd} = -\delta^{cd}_{ab}$) and T_{h_i} are the generators of the coset algebra branched with respect to H.

In our example we have $h_1 = i = \{1,2\}$, $h_2 = s = \{3,4\}$, $h_3 = \{5\}$ and using (2.13)–(2.14) and (3.29)–(3.30), the M matrices are

$$\begin{aligned} M^{ij} &= -\left(c - \frac{a^2}{4c}\right) V^5 \epsilon^{ij} \,, \qquad M^{5i} = \frac{a^2}{4c} \epsilon^{ij} V_j \,, \\ M^{st} &= \left(c - \frac{a^2}{4c}\right) V^5 \epsilon^{st} \,, \qquad M^{5s} = -\frac{a^2}{4c} \epsilon^{st} V_t \,, \\ & \qquad\qquad M^{is} = 0 \,. \end{aligned} \tag{3.34}$$

and the covariant derivatives turn out to be

$$\begin{aligned} \mathcal{D}_i &= \left(-a T_i - \frac{a^2}{2c} \epsilon_i{}^j T_{5j}\right) , \\ \mathcal{D}_s &= \left(-a T_s + \frac{a^2}{2c} \epsilon_s{}^t T_{5t}\right) , \\ \mathcal{D}_5 &= \left(-c T_5 - 2\left(c - \frac{a^2}{4c}\right)(T_{12} - T_{34})\right) . \end{aligned} \tag{3.35}$$

Acting on a given harmonic $Y^{\{\nu\}m}_{i_\xi}(y)$, identified by the irrep $\{\nu\}$ of G, with the covariant derivative, gives

$$\mathcal{D}_{h_i} Y^{\{\nu\}m}_{h_j} = -r(i) \left[\mathbb{D}(T_{h_i})\right]_{h_j h_k} Y^{\{\nu\}m}_{h_k} + M^{ab}_{a_i} \left[\mathbb{D}(T_{h_i})\right]_{h_j h_k} Y^{\{\nu\}m}_{h_k} \,. \tag{3.36}$$

Note that the second term of the right hand side of (3.36) acts on the harmonics as an element of the SO(d) Lie algebra, while the first term acts as an element of the Lie algebra of G, the isometry group.

Let us now observe that the action of $\mathcal{D}_{h_i}$ on any harmonic is perfectly known once we define it on the basic harmonic $L^{-1}(y)$ in the fundamental representation of G (if G is an orthogonal group it is useful to consider the action of $\mathcal{D}_{h_i}$ on the fundamental and spinor representations). On the basis of this observation, the evaluation of the first term is usually done by identifying $\{\nu\}$ with a Young tableaux $[\lambda_1, \ldots, \lambda_{[d/2]}]$. The action of $\mathbb{D}(T_{h_i})$ on such a tableaux is obtained by tensoring the action of the fundamental representation of T_{h_i} on each box of the tableaux

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
i_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & i_{\lambda_1} \\
\hline
j_1 & \cdots & \cdots & \cdots & \cdots & j_{\lambda_2} \\
\cline{1-6}
\vdots & \vdots & & \vdots \\
\cline{1-4}
k_1 & \cdots & k_{\lambda_{[d/2]}} \\
\cline{1-3}
\end{array} \tag{3.37}$$

and performing the required (anti-)symmetrisations. Note that since in the fundamental representation

$$\mathbb{D}(T_{h_i})_k{}^j \, \boxed{j} = c_i \, \boxed{k} \,, \tag{3.38}$$

T_{h_i} can be thought as an operator destroying $\boxed{j}$ and creating $\boxed{k}$ times a numerical coefficient (see the explicit examples of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ in the sequel).

The evaluation of the second term in (3.36) is simpler: it is sufficient to use the matrix realization of T_{ab} in the $[\lambda_1, \ldots, \lambda_{[d/2]}]$ representation of $\mathrm{SO}(d)$.

Actually we are interested in the second order and first order Laplace-Beltrami operators given in (3.2). The first order operators can be written in terms of

$$\gamma^a \mathcal{D}_a \equiv \sum_{i=1}^{N} \gamma^{a_i} \mathcal{D}_{a_i} = \sum_{i=1}^{N} \gamma^{a_i} (-r(i) T_{a_i} + M_{a_i}^{ab} T_{ab}) \,, \tag{3.39}$$

while the second order operators are all given, apart from curvature terms (which are known from the G/H geometry), in terms of the covariant laplacean, namely:

$$\begin{aligned}
\Box = \mathcal{D}_a \mathcal{D}^a \equiv \sum_i \mathcal{D}_{a_i} \mathcal{D}^{a_i} &= \sum_i (-r(i) T_{a_i} + M_{a_i}^{ab} T_{ab})(-r(i) T^{a_i} + M^{ab\ a_i} T_{ab}) = \\
&= \sum_i \Big(r(i)^2 T_{a_i} T^{a_i} - 2 r(i) T_{a_i} M^{ab\ a_i} T_{ab} + M_{a_i}^{ab} M^{cd\ a_i} T_{ab} T_{cd} \Big) \,.
\end{aligned} \tag{3.40}$$

On an $\mathrm{SO}(d)$ scalar harmonic which, being a $\mathrm{SO}(d)$ singlet, is also an H-singlet, the previous expression assumes the remarkably simple form

$$\Box Y_0^{\{\nu\}m} \equiv \mathcal{D}_a \mathcal{D}^a Y_0^{\{\nu\}m} = \sum_i r(i)^2 T_{a_i} T^{a_i} Y_0^{\{\nu\}m} \,, \tag{3.41}$$

where $Y_0^{\{\nu\}m}$ is the one-dimensional $\mathrm{SO}(d)$ *and* H-singlet.

Formulae (3.39)–(3.41) can be directly used in our example by setting $a_1 = \{1,2\} \equiv \{i\}$, $a_2 = \{3,4\} \equiv \{s\}$, $a_3 = \{5\}$, $r(i) = 6\,e^2$, $r(s) = 6\,e^2$, $r(5) = \frac{9}{2}\,e^2$, identifying the T_{ab} with the $\mathrm{SO}(5)$ generators and using the values of $M_{a_i}^{ab} = \{M_i^{ab}, M_s^{ab}, M_5^{ab}\}$ given in equations (3.34).

Let us now show how to compute the spectrum of the physical masses for the AdS fields appearing in the expansion (3.9). This is equivalent to compute the eigenvalues of the Laplace-Beltrami operators introduced before, the physical masses being the eigenvalues or simply related to them.

To compute the eigenvalues of a generic operator $\boxtimes$ we can compute either the explicit $\dim[\lambda] \times \dim[\lambda]$ numerical matrix obtained from the expression of the covariant laplacean plus curvature terms obtained from (3.2) after the action of the operators T_{a_i}, T_{ab} in the representation $\{\alpha_i\}$ and $[\lambda]$ on the given harmonic has been evaluated; or, what happens to be easier in practice, one can think of (3.39) or (3.40) as a $\dim[\lambda] \times \dim[\lambda]$ matrix whose entries contain the operators T_{a_i} and/or $T_{a_i}T^{a_i}$. In that case we write:

$$Y^{\{\nu\}m}_{ab\ldots} = \begin{pmatrix} Y_{i_1} \\ \vdots \\ Y_{i_N} \end{pmatrix}^{\{\nu\}m} \tag{3.42}$$

and applying $\boxtimes_y$ to both sides of (3.9) (suppressing for a moment the indices $\{\nu\}m$ since they remain inert in the computation) we have

$$\boxtimes\Phi^{[\lambda]}_{ab\ldots}(x,y) = \boxtimes\left\{\sum_{\xi=1}^{N}\Phi_{i_\xi}\begin{pmatrix} 0 \\ \vdots \\ Y_{i_\xi} \\ \vdots \\ 0 \end{pmatrix}\right\} = \sum_{\xi=1}^{N}\Phi_{i_\xi}\,\boxtimes_{\xi\xi'}\begin{pmatrix} 0 \\ \vdots \\ Y_{i_{\xi'}} \\ \vdots \\ 0 \end{pmatrix}. \tag{3.43}$$

Recalling that $\boxtimes_{\xi\xi'}$ is a matrix of "operators" acting on the fragments Y_{i_ξ} we have

$$\boxtimes_{\xi\xi'}Y_{i_{\xi'}} = M_{\xi\xi'}Y_{i_{\xi'}}\,, \tag{3.44}$$

where $M_{\xi\xi'}$ is now a numerical entry. Thus

$$\boxtimes Y_{ab\ldots}(x,y) = \begin{pmatrix} \Phi_{i_1}M_{11}Y_{i_1} & \ldots & \Phi_{i_1}M_{1N}Y_{i_N} \\ \vdots & & \vdots \\ \Phi_{i_N}M_{N1}Y_{i_1} & \ldots & \Phi_{i_N}M_{NN}Y_{i_N} \end{pmatrix}. \tag{3.45}$$

By diagonalisation of $M_{\xi\xi'}$ we finally find

$$\boxtimes\Phi^{\{\nu\}}_{ab\ldots}(x,y) = \sum_{\{\nu\}m}\begin{pmatrix} \lambda_1\Phi_{i_1}Y_{i_1} \\ \vdots \\ \lambda_N\Phi_{i_N}Y_{i_N} \end{pmatrix}^{\{\nu\}m} = \sum_{\{\nu\}m}\sum_{\xi=1}^{N}\Phi^{\{\nu\}m}_{i_\xi}\lambda_\xi\begin{pmatrix} 0 \\ \vdots \\ Y^{\{\nu\}m}_{i_\xi} \\ \vdots \\ 0 \end{pmatrix} \tag{3.46}$$

and we see that in general we have different eigenvalues λ_ξ in the different subspaces $\{\beta_i\}$.

3.3 Harmonic analysis on T^{11}

Having explained the general procedure in the previous section, let us now restrict our attention to the harmonic analysis on T^{11}.

As we discussed in section 3.1, the first thing to do is to consider the background solution generating the $AdS_5 \times T^{11}$ geometry and compute the fluctuations of the fields around this solution. This latter is given by the following values of the ten-dimensional fields [26]

$$\begin{aligned} F_{abcde} &= e\epsilon_{abcde}\,, & R^a{}_b &= 2e^2\delta^a_b\,, \\ F_{mnpqr} &= -e\epsilon_{mnpqr}\,, & R^m{}_n &= -2e^2\delta^m_n\,, \\ B = A_{MN} &= 0, & \psi_M &= \chi = 0\,, \end{aligned} \tag{3.47}$$

where F_{abcde} and F_{mnpqr} are the projections on T^{11} and AdS_5 of the ten-dimensional five-form F defined as $F = dA_4$, A_4 being the real self-dual four-form of type IIB supergravity. The other fields of type IIB supergravity are: the metric $G_{MN}(x,y)$ with internal and space-time components $g_{ab}(y)$, $g_{\mu\nu}(x)$ whose Ricci tensors in this background are given in (3.47) and the complex 0-form and 2-form B and A_{MN} (the fermionic fields ψ_M and χ are obviously zero in the background (3.47)).

To determine the KK modes, one has first to determine the linearised field equations around the background (3.47) and then expand the excitations into the T^{11} harmonics. As explained in [12, 29], the general expansion can be simplified by choosing covariant gauge conditions in the internal space.

We are not going to give a detailed derivation of the linearised equations or gauge-fixing conditions for our case. The right procedure has been explained in [29] to which we refer. There are however some subtleties and differences arising in making the same computations in our case which will be examined below. To be more precise, while for most of the field excitations we can repeat the KK expansion and impose the gauge-fixing conditions as reported in [29], we have to be careful for the expansion of the modes deriving from the fourth-rank antisymmetric tensor A_{MNPQ}.

We name the ten-dimensional and five-dimensional fields as in [29] and we report in the table 1 the relevant notations.

Using these notations the expansion of the fields, except A_{MNPQ}, becomes

$$\begin{aligned} h'_{\mu\nu}(x,y) &= \sum_{\{\nu\}} H^{\{\nu\}}_{\mu\nu}(x)Y^{\{\nu\}}(y)\,, & A_{\mu\nu}(x,y) &= \sum_{\{\nu\}} a^{\{\nu\}}_{\mu\nu}(x)Y^{\{\nu\}}(y)\,, \\ h_{\mu a}(x,y) &= \sum_{\{\nu\}} B^{\{\nu\}}_{\mu}(x)Y^{\{\nu\}}_a(y)\,, & A_{\mu a}(x,y) &= \sum_{\{\nu\}} a^{\{\nu\}}_{\mu}(x)Y^{\{\nu\}}_a(y)\,, \\ h_{(ab)}(x,y) &= \sum_{\{\nu\}} \phi^{\{\nu\}}(x)Y^{\{\nu\}}_{(ab)}(y)\,, & A_{ab}(x,y) &= \sum_{\{\nu\}} a^{\{\nu\}}(x)Y^{\{\nu\}}_{[ab]}(y)\,, \\ h^a{}_a(x,y) &= \sum_{\{\nu\}} \pi^{\{\nu\}}(x)Y^{\{\nu\}}(y)\,, & B(x,y) &= \sum_{\{\nu\}} B^{\{\nu\}}(x)Y^{\{\nu\}}(y)\,. \end{aligned} \tag{3.48}$$

Dim	fields					harm.
10 D	$h'_{\mu\nu}(x,y)$	$h^a{}_a(x,y)$	$A_{abcd}(x,y)$	$B(x,y)$	$A_{\mu\nu}(x,y)$	
5 D	$H_{\mu\nu}(x)$	$\pi(x)$	$b(x)$	$B(x)$	$a_{\mu\nu}(x)$	$Y(y)$
10 D	$h_{a\mu}(x,y)$	$A_{\mu abc}(x,y)$	$A_{\mu a}(x,y)$			
5 D	$B_\mu(x)$	$\phi_\mu(x)$	$a_\mu(x)$			$Y_a(y)$
10 D	$A_{\mu\nu ab}(x,y)$	$A_{ab}(x,y)$				
5 D	$b^{\pm}_{\mu\nu}(x)$	$a(x)$				$Y_{[ab]}(y)$
10 D	$h_{ab}(x,y)$					
5 D	$\phi(x)$					$Y_{(ab)}(y)$
10 D	$\lambda(x,y)$	$\psi_{(a)}(x,y)$	$\psi_\mu(x,y)$			
5 D	$\lambda(x)$	$\psi^{(L)}(x)$	$\psi_\mu(x)$			$\Xi(y)$
10 D	$\psi_a(x,y)$					
5 D	$\psi^{(T)}(x)$					$\Xi_a(y)$

Table 1: Fields appearing in the harmonic expansion.

Note that in (3.48) do not appear derivative terms in the harmonics; this is due to the fact that using the gauge invariance freedom in the internal space, which we then fix by imposing

$$\begin{aligned} \mathcal{D}^a h_{(ab)} &= \mathcal{D}^a h_{a\mu} = 0\,, \\ \mathcal{D}^a A_{ab} &= \mathcal{D}^a A_{a\mu} = 0\,, \end{aligned} \tag{3.49}$$

we can restrict ourselves to the transverse harmonics, namely harmonics satisfying the condition $\mathcal{D}^a Y_{ab\ldots} = 0$.

In an analogous way, in the harmonic expansion of the fourth-rank antisymmetric tensor fluctuations (denoted by a_{MNPQ}), we can choose the covariant gauge conditions

$$\mathcal{D}^a a_{abcd} = \mathcal{D}^a a_{abc\mu} = \mathcal{D}^a a_{ab\mu\nu} = \mathcal{D}^a a_{a\mu\nu\rho} = 0\,, \tag{3.50}$$

and remove again terms with gradients from the harmonics of the various fluctuations. This means that we can expand these fluctuations in KK modes (we use the collective index $\{\nu\}$ to denote the doublet (j,l))

$$a_{\mu\nu\rho\sigma}(x,y) = \sum_{\{\nu\}} b^{\{\nu\}}_{\mu\nu\rho\sigma}(x) Y^{\{\nu\}}(y)\,, \tag{3.51a}$$

$$a_{\mu\nu\rho a}(x,y) = \sum_{\{\nu\}} b^{\{\nu\}}_{\mu\nu\rho}(x) Y^{\{\nu\}}_a(y)\,, \tag{3.51b}$$

$$a_{\mu\nu ab}(x,y) = \sum_{\{\nu\}} b^{\{\nu\}}_{\mu\nu}(x) Y^{\{\nu\}}_{ab}(y)\,, \tag{3.51c}$$

$$a_{\mu abc}(x,y) = \sum_{\{\nu\}} \phi_\mu^{\{\nu\}}(x) Y_{abc}^{\{\nu\}}(y)\,, \tag{3.51d}$$

$$a_{abcd}(x,y) = \sum_{\{\nu\}} b^{\{\nu\}}(x) Y_{abcd}^{\{\nu\}}(y)\,. \tag{3.51e}$$

We can achieve further simplifications if we consider the duality conditions in a five-dimensional space and the gauge conditions (3.50). From $\mathcal{D}^a a_{abcd} = 0$, equation (3.51e) and since in five dimensions a 4-form is dual to a 1-form, we have

$$\epsilon^{abcde}\mathcal{D}_d Y_e^{\{\nu\}} = 0\,, \tag{3.52}$$

Now we recall that since the topology of T^{11} is that of $S^2 \times S^3$ its Betti numbers can be easily derived through the Künneth formula from the product of those of the spheres in the product. This tells us that there are no non-trivial one-cycles and so no closed non-exact 1-forms, exactly as in the case of the S^5 compactification of [29]. Therefore (3.52) implies that Y_e is an exact 1-form. Thus we can simplify the a_{abcd} expansion to

$$a_{abcd} = \sum_{\{\nu\}} b^{\{\nu\}} \epsilon_{abcd}{}^e \mathcal{D}_e Y^{\{\nu\}}\,. \tag{3.53}$$

One could try to repeat the same procedure for the $a_{\mu abc}$ field, but here comes the point. While this procedure is straightforward for the S^5 internal space analyzed in [29], this is not the case for the T^{11} manifold. From the gauge condition $\mathcal{D}^a a_{\mu abc} = 0$, equation (3.51d) and the duality relations, we find

$$\epsilon^{abcde}\mathcal{D}_c Y_{de}^{\{\nu\}} = 0\,, \tag{3.54}$$

but now Y_{de} can be a closed non-exact two-form since the second and third Betti numbers of T^{11} are different from zero and in fact

$$b_2 = b_3 = 1\,. \tag{3.55}$$

This means that there exist a non-exact closed two-form on this space which can satisfy (3.54).

In [30] it has been shown that this single Betti 2-form must be a G-singlet and thus there is only the $\{\nu\} = \{G\text{-singlet}\} = \{0\}$ mode which cannot be treated as in [29]. The $a_{\mu abc}$ expansion for $AdS_5 \times T^{11}$ is therefore

$$a_{\mu abc} = \sum_{\{\nu\}} \left[\phi_\mu^{\{\nu\}} \epsilon_{abc}{}^{de} \mathcal{D}_d Y_e^{\{\nu\}}\right] + \tilde{\phi}_\mu^{(0)} \epsilon_{abc}{}^{de} Y_{de}^{(0)}\,, \tag{3.56}$$

which means that the only difference with the expansion in [29] is the last term in (3.56).

Let us then analyze the consequences of this modification on the linearised equations of motion. It is easy to see that the only equations which are affected by this change are the (E2), (M2) and (M3) equations of [29], namely

$$\begin{aligned}
&\frac{1}{2}(\Box_x+\Box_y)h_{\mu a}-\frac{1}{2}\mathcal{D}_\mu\mathcal{D}^\rho h_{a\rho}-\\
&-\frac{1}{2}\mathcal{D}_a\mathcal{D}^\rho h'_{\mu\rho}-\frac{4}{15}\mathcal{D}_\mu\mathcal{D}_a h_\gamma^\gamma+\\
+\frac{1}{2}\mathcal{D}_\mu\mathcal{D}_a h'^\sigma_\sigma-\frac{1}{2}\mathcal{D}_\mu\mathcal{D}^b h_{(ab)}-\frac{1}{2}\mathcal{D}_a\mathcal{D}^b h_{\mu b}&=-\frac{e}{6}\epsilon_\mu{}^{\nu\rho\sigma\tau}(\partial_a a_{\nu\rho\sigma\tau}-4\partial_\nu a_{a\rho\sigma\tau})-\\
&-\frac{e}{6}\epsilon_a{}^{bcde}(\partial_\mu a_{bcde}-4\partial_b a_{\mu cde}) \qquad (3.57)\\
\partial_a a_{\mu\nu\rho\sigma}+4\partial_{[\mu}a_{\nu\rho\sigma]a}&=\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma a}{}^{\tau bcde}(\partial_\tau a_{bcde}+4\partial_b a_{cde\tau})+\\
&+e\epsilon_{\mu\nu\rho\sigma}{}^\tau h_{a\tau} \qquad (3.58)\\
3\partial_{[\mu}a_{\nu\rho]ab}+2\partial_{[a}a_{b]\mu\nu\rho}&=\frac{10}{5!}\epsilon_{\mu\nu\rho ab}{}^{\sigma\tau cde}(3\partial_\gamma a_{de\sigma\tau}+2\partial_\sigma a_{\tau\gamma de}) \qquad (3.59)
\end{aligned}$$

The (3.57) and (3.59) equations now contain extra terms of the form

$$\epsilon_a{}^{bcde}\partial_b a_{\mu cde}=\tilde{\phi}^0\epsilon_a{}^{bcde}\partial_b\epsilon_{cde}{}^{fg}Y^0_{fg}\sim\tilde{\phi}^0\partial^b Y^0_{ab}\,, \qquad (3.60)$$

which therefore reduce to zero due to the transversality condition on Y_{ab} ($\mathcal{D}^a Y_{ab}=0$) deriving from the gauge conditions (3.50) applied to (3.51c).

We are left with the (3.58) equation which, after insertion of the KK modes using (3.48)–(3.51), contains in the r.h.s. a new term in the following sector [29, equation (M2.2)]

$$3\partial_{[\mu}b^0_{\nu\rho]}-\frac{\Delta}{4}\epsilon_{\mu\nu\rho}{}^{\sigma\tau}b^0_{\sigma\tau}=2\epsilon_{\mu\nu\rho}{}^{\sigma\tau}\mathcal{D}_\sigma\tilde{\phi}^0_\tau\,, \qquad (3.61)$$

where Δ is the eigenvalue of the first-order operator $\star d$ on the two-form (3.3) and all the fields are in the singlet representation of the isometry group. The divergence $\mathcal{D}^\mu$ of this equation makes it independent on the new $\tilde{\phi}^0$ field due to the obvious identity $\mathcal{D}_{[\mu}\mathcal{D}_\nu\tilde{\phi}^0_{\rho]}=0$, while we find a new equation of motion for the $\tilde{\phi}^0$ field by contraction of (3.61) with $\epsilon^{\mu\nu\rho}{}_{\sigma\tau}\mathcal{D}^\sigma$. This yields

$$\mathcal{D}^\sigma\mathcal{D}_{[\sigma}\tilde{\phi}^0_{\tau]}=\mathcal{D}^\sigma b^0_{\sigma\tau}\,. \qquad (3.62)$$

Eq. (3.62) is easily seen to correspond to a new massless vector. Indeed, since b^0_{mn} is massive, i.e.

$$\epsilon_{\mu\nu}{}^{\rho\sigma\tau}\mathcal{D}_\rho b^0_{\sigma\tau}=m_0 b^0_{\mu\nu}\,, \qquad (3.63)$$

its divergence vanishes $\mathcal{D}^s b_{st}=0$, and (3.62) becomes

$$\mathcal{D}^\sigma\mathcal{D}_{[\sigma}\tilde{\phi}^0_{\tau]}=0\,. \qquad (3.64)$$

We see that the presence of non-trivial homology cycle on our manifold implies the presence of a massless vector in the singlet representation of the isometry group, and if supersymmetry is present, of an entire vector multiplet named Betti multiplet in [23].

Now that we have discussed the linearised equations of motion, we are in position to proceed to perform the harmonic expansion and thus to determine which irreducible representations $\{\nu\}$ of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ do occur in (3.48)–(3.51).

As explained in the general setting of section 2, in order to answer this question a preliminary step is to analyze the branching of the SO(5) representations of the fields appearing in the l.h.s. of (3.23) into $\mathrm{U}_H(1)$ representations, according to the general formulae (3.17)–(3.19)

The internal SO(5) representations we need to branch are:

$$\begin{array}{lll}
[0,0] & \text{for the fields } h'_{\mu\nu}, h^a{}_a, A_{\mu\nu}, B & (\underline{1})\,, \\
{[1,0]} & \text{for the fields } h_{\mu a}, A_{\mu a}, A_{\mu\nu\rho a}, A_{abcd} & (\underline{5})\,, \\
{[1,1]} & \text{for the fields } A_{ab}, A_{\mu\nu ab}, A_{\mu abc} & (\underline{10})\,, \\
{[2,0]} & \text{for the fields } h_{(ab)} & (\underline{14})\,, \\
\left[\frac{1}{2},\frac{1}{2}\right] & \text{for the fields } \lambda, \psi^\alpha_\mu & (\underline{4})\,, \\
\left[\frac{3}{2},\frac{1}{2}\right] & \text{for the fields } \psi^\alpha_a & (\underline{16})
\end{array}$$

where on the extreme r.h.s. we have written the corresponding dimensions.

The explicit example worked out in (3.21), (3.22) do already give the branching for the irrepses $\underline{5}$ and $\underline{4}$, namely

$$\begin{array}{ll}
\mathbf{5} \to 1 \oplus -1 \oplus 1 \oplus -1 \oplus 0 & [\lambda_1,\lambda_2] = [1,0], \\
\mathbf{4} \to 1 \oplus -1 \oplus 0 \oplus 0 & [\lambda_1,\lambda_2] = \left[\frac{1}{2},\frac{1}{2}\right].
\end{array} \tag{3.65}$$

where we have named the $\mathrm{U}_H(1)$ irrepses by their charge.

From (3.65) we easily find the analogous breaking law for antisymmetric tensors ($[\lambda_1,\lambda_2] = [1,1]$), symmetric traceless tensors ($[\lambda_1,\lambda_2] = [2,0]$) and spin tensors ($[\lambda_1,\lambda_2] = [3/2,1/2]$) by taking suitable combinations:

$$\begin{array}{ll}
\mathbf{10} \to \pm1 \oplus \pm1 \oplus \pm2 \oplus 0 \oplus 0 \oplus 0 \oplus 0 & [\lambda_1,\lambda_2] = [1,1], \\
\mathbf{16} \to \pm2 \oplus \pm2 \oplus \pm1 \oplus \pm1 \oplus \pm1 \oplus \pm1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 & [\lambda_1,\lambda_2] = \left[\frac{3}{2},\frac{1}{2}\right], \\
\mathbf{14} \to \pm2 \oplus \pm2 \oplus \pm2 \oplus \pm1 \oplus \pm1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 & [\lambda_1,\lambda_2] = [2,0].
\end{array} \tag{3.66}$$

As it has been stressed in the sentence after (3.25), we know that the only irrepses $\{\nu\}$ of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ which appear in the harmonic expansion of a field belonging to the $\mathrm{SO}(d)$ irrep $[\lambda]$, are those which, when branched with respect to $\mathrm{U}_H(1)$, contain at least one of the fragments of the branching (3.65) or (3.66).

To select such representations, we recall that a generic G tableaux can be written as (3.12)

$$\underbrace{\boxed{1}\boxed{\cdots}\boxed{1}}_{m_1}\underbrace{\boxed{2}\boxed{\cdots}\boxed{2}}_{m_2}\otimes\underbrace{\boxed{1}\boxed{\cdots}\boxed{1}}_{n_1}\underbrace{\boxed{2}\boxed{\cdots}\boxed{2}}_{n_2}$$

and that we have the (3.15) relations

$$\begin{aligned} q + r &\equiv m_2 - m_1\,, \\ q - r &\equiv n_2 - n_1 . \end{aligned}$$

We observe that as long as $m_2 - m_1$ and $n_2 - n_1$ are even or odd, the same is true for $m_1 + m_2$ and $n_1 + n_2$. Therefore the parity of $2j$ and $2l$ is the same as that of $2j_3$ and $2l_3$ and since $2j_3 + 2l_3 = 2q$ can be even or odd, the same is true for $2j + 2l$. Thus j and l must either be both integers or both half-integers. This means that the q value of any $\mathrm{U}_H(1)$ fragment of the SO(5) fields is always contained in any SO(5)-harmonic in the irrep (j, l) provided that j and l are both integers or half-integers. Since $q + r$ and $q - r$ are related to the third component of the "angular momentum" of the two SU(2) factors, one also has the conditions $|q + r| \le 2j$ and $|q - r| \le 2l$. The two above conditions select the harmonics appearing in the expansion.

4. Computing the spectrum

The scalar harmonic. The spectrum of the scalar harmonics $Y^{(j,l)}_{[0,0]} = Y^{(j,l,r)}_{q=0}$ is easily computed, since the relevant five-dimensional invariant operator is simply the covariant laplacean (3.2a):

$$\Box = \mathcal{D}^a\mathcal{D}_a \equiv \mathcal{D}^i\mathcal{D}_i + \mathcal{D}^s\mathcal{D}_s + \mathcal{D}^5\mathcal{D}_5\,. \tag{4.1}$$

Using (3.41)

$$\Box Y^{(j,l,r)}_{q=0} = (-a^2(T_iT_i + T_sT_s) - c^2T_5T_5)Y^{(j,l,r)}_{q=0}\,, \tag{4.2}$$

In order to evaluate (4.2), we set

$$\begin{aligned} T_i &= -\frac{i}{2}\sigma_i\,, \qquad T_s = -\frac{i}{2}\hat\sigma_s\,, \\ T_5 &= T_3 - \hat T_3 = \frac{i}{2}(\hat\sigma_3 - \sigma_3)\,, \end{aligned} \tag{4.3}$$

where σ_A and $\hat\sigma_A$ are ordinary Pauli matrices. Using the relations

$$\begin{array}{lll} \sigma_1\boxed{1} = \boxed{2} & \sigma_2\boxed{1} = -i\boxed{2} & \sigma_3\boxed{1} = \boxed{1} \\ \sigma_1\boxed{2} = \boxed{1} & \sigma_2\boxed{2} = i\boxed{1} & \sigma_3\boxed{2} = -\boxed{2} \end{array} \tag{4.4}$$

(the same is true for $\hat\sigma$) and observing that on a Young tableaux the σ's act like a derivative (Leibnitz rule), we find on the first tableaux of (3.12)

$$
\begin{aligned}
(\sigma_1\sigma_1 + \sigma_2\sigma_2)\begin{array}{|c|c|c|}\hline \ & \cdots & \ \\ \hline\end{array} &= (2m_1(m_2+1) + 2m_2(m_1+1))\begin{array}{|c|c|c|}\hline \ & \cdots & \ \\ \hline\end{array} \\
&= 4(j(j+1) - (j_3)^2)\begin{array}{|c|c|c|}\hline \ & \cdots & \ \\ \hline\end{array}\,.
\end{aligned} \tag{4.5}
$$

An analogous result holds when acting with $\hat\sigma_1\hat\sigma_1 + \hat\sigma_2\hat\sigma_2$ on the second tableaux of (3.12), with $j \leftrightarrow l$.

The eigenvalue of $(\hat\sigma_3 - \sigma_3)^2$ on (3.12) is

$$
(m_2 - m_1 + n_2 - n_1)^2 = 4(j_3 + l_3)^2\,. \tag{4.6}
$$

For a scalar, $q = 0$ and so, from (3.15), we have

$$
j_3 = -l_3 = \frac{r}{2}\,, \tag{4.7}
$$

and we find

$$
\Box Y^{(j,l,r)}_{(0)} = \left[a^2 j(j+1) + b^2 l(l+1) + (4c^2 - a^2 - b^2)\frac{r^2}{4}\right] Y^{(j,l,r)}_{(0)}\,. \tag{4.8}
$$

Substituting the values of a,b and c given in (2.19), we obtain

$$
\Box Y^{(j,l,r)}_{(0)} = H_0(j,l,r) Y^{(j,l,r)}_{(0)}\,, \tag{4.9}
$$

where

$$
H_0(j,l,r) \equiv 6\left(j(j+1) + l(l+1) - \frac{r^2}{8}\right) \tag{4.10}
$$

is the eigenvalue of the laplacean. The same result was first given in [4] using differential methods and it was also obtained in [5].

As shown in table 1, the scalar harmonic appears in the expansion of the ten-dimensional fields $h'_{\mu\nu}(x,y)$, $B(x,y)$, $h^a{}_a(x,y)$, $A_{abcd}(x,y)$ and $A_{\mu\nu}$. The masses of the corresponding five-dimensional space-time fields are thus given in terms of $H_0(j,l,r)$, and explicitly they read

$$
\begin{aligned}
m^2(H_{\mu\nu}) &= H_0\,, && (4.11)\\
m^2(B) &= H_0\,, && (4.12)\\
m^2(\pi,b) &= H_0 + 16 \pm 8\sqrt{H_0+4}\,, && (4.13)\\
m^2(a_{\mu\nu}) &= 8 + H_0 \pm 4\sqrt{H_0+4}\,. && (4.14)
\end{aligned}
$$

Note that while the laplacean acts diagonally on the AdS_5 fields $H_{\mu\nu}(x)$ and $B(x)$, the eigenvalues for $\pi(x)$ and $b(x)$, which appear entangled in the linearised equations of motion [25, 29], have been obtained after diagonalisation of a two by two matrix.

JHEP11(1999)009

The spinor harmonic. We now give the action of the $\not{\mathcal{D}}$ operator (3.2e) on the spinor representation of SO(5). Equation (3.35) yields

$$\not{\mathcal{D}} = \gamma^a \mathcal{D}_a = \gamma^i \left(-aT_i - \frac{a^2}{2c} \epsilon_{ij} T_5{}^j \right) + \gamma^s \left(-aT_s + \frac{a^2}{2c} \epsilon_{st} T_5{}^t \right) + \\ + \gamma^5 \left(-cT_5 - 2\left(c - \frac{a^2}{4c} \right) (T_{12} - T_{34}) \right), \tag{4.15}$$

where T_{ab} are the SO(5) generators in the spinor representation. A straightforward computation gives

$$\not{\mathcal{D}} = \begin{pmatrix} icT_5 \mathbb{1}_2 + \left(\frac{a^2}{4c} + c \right) \sigma^3 & -a \left(\sigma^i T_i + \sigma^3 \hat{T}_1 - i\mathbb{1}_2 \hat{T}_2 \right) \\ a\left(\sigma^i T_i + \sigma^3 \hat{T}_1 + i \mathbb{1}_2 \hat{T}_2 \right) & -icT_5 \mathbb{1}_2 \end{pmatrix}. \tag{4.16}$$

When substituting the values of c and a in the matrix (4.16) we note that (2.19) defines them only up to a sign. However, only one of them is consistent with supersymmetry. Indeed, if a complex Killing spinor $\eta(y)$ generating $\mathcal{N}=2$ supersymmetry in AdS_5 is to exist, it must have the form

$$\eta = \begin{pmatrix} k \\ l \\ 0 \\ 0 \end{pmatrix}, \qquad k, l \in \mathbb{C} \tag{4.17}$$

since, being an SU(2) $\times$ SU(2) singlet, it must satisfy $T_H \eta = 0$ (see (3.22)). At this point the Killing equation $\not{\mathcal{D}} \eta = \frac{5}{2} e \eta$ can be computed from (4.16) observing that on an SU(2) $\times$ SU(2) singlet the T_a generators have a null action and thus, using (4.17),

$$\not{\mathcal{D}} \eta = \begin{pmatrix} \left(\frac{a^2}{4c} + c \right) \sigma^3 & 0 \\ 0 & 0 \end{pmatrix} \eta = \frac{5}{2} e \, \eta \,. \tag{4.18}$$

This gives the correct value only if we choose $l = 0$ and

$$c = -\frac{3}{2} e \,, \tag{4.19}$$

while the sign of $a = \pm\sqrt{6}\, e$ is unessential.

Recalling the meaning of c as the rescaling of the vielbein V^5, *we conclude that* T^{11} *admits a Killing spinor, leading to* $\mathcal{N}=2$ *supersymmetry on* AdS_5*, only for one orientation of* T^{11}.

In order to compute the mass eigenvalues, we write (4.16) as an explicit 4×4 matrix whose entries are operators, according to the discussion given at the end of section 3.2

$$\not{\mathcal{D}} = e \begin{pmatrix} -i\frac{3}{2} T_5 + \frac{5}{2} & 0 & \sqrt{6} \hat{T}_+ & \sqrt{6} T_- \\ 0 & -i\frac{3}{2} T_5 - \frac{5}{2} & \sqrt{6} T_+ & -\sqrt{6} \hat{T}_- \\ -\sqrt{6} \hat{T}_- & -\sqrt{6} T_- & \frac{3}{2} i T_5 & 0 \\ -\sqrt{6} T_+ & \sqrt{6} \hat{T}_+ & 0 & \frac{3}{2} i T_5 \end{pmatrix}, \tag{4.20}$$

where we have set

$$T_{\pm} \equiv T_1 \pm iT_2, \qquad \hat{T}_{\pm} \equiv \hat{T}_1 \pm i\hat{T}_2 . \tag{4.21}$$

Note that (4.20) acts on the four-dimensional spinor representation of the SO(5) spinor harmonic

$$\Xi = \begin{pmatrix} Y^{(j,l,r-1)}_{(0)} \\ Y^{(j,l,r+1)}_{(0)} \\ Y^{(j,l,r)}_{(-1)} \\ Y^{(j,l,r)}_{(+1)} \end{pmatrix},$$

which has been decomposed in one-dimensional $\mathrm{U}_H(1)$ fragments identified by their charge according to (3.22). The operatorial matrix (4.20) can now be replaced by a numerical one $M_{\xi\xi'}$ according to the discussion of sect 2, which is simply obtained from (4.20) by substituting in each entry the value of the T-operators on the harmonics. The fundamental substitutions one has to make are

$$\begin{aligned}
T_+ Y^{(j,l,r)}_{(q)} &= -i\left(j - \frac{q+r}{2}\right) Y^{(j,l,r+1)}_{(q+1)}, \\
T_- Y^{(j,l,r)}_{(q)} &= -i\left(j + \frac{q+r}{2}\right) Y^{(j,l,r-1)}_{(q-1)}, \\
\hat{T}_+ Y^{(j,l,r)}_{(q)} &= -i\left(l - \frac{q-r}{2}\right) Y^{(j,l,r-1)}_{(q+1)}, \\
\hat{T}_- Y^{(j,l,r)}_{(q)} &= -i\left(l + \frac{q-r}{2}\right) Y^{(j,l,r+1)}_{(q-1)}, \\
T_5 Y^{(j,l,r)}_{(q)} &= ir Y^{(j,l,r)}_{(q)} .
\end{aligned} \tag{4.22}$$

From this action we obtain

$$\begin{pmatrix}
1+\frac{3}{2}r & 0 & -i\sqrt{6}\left(l+\frac{r}{2}+\frac{1}{2}\right) & -i\sqrt{6}\left(j+\frac{r}{2}+\frac{1}{2}\right) \\
0 & -1+\frac{3}{2}r & -i\sqrt{6}\left(j-\frac{r}{2}+\frac{1}{2}\right) & i\sqrt{6}\left(l-\frac{r}{2}+\frac{1}{2}\right) \\
i\sqrt{6}\left(l-\frac{r}{2}+\frac{1}{2}\right) & i\sqrt{6}\left(j+\frac{r}{2}+\frac{1}{2}\right) & -\frac{3}{2}r & 0 \\
i\sqrt{6}\left(j-\frac{r}{2}+\frac{1}{2}\right) & -i\sqrt{6}\left(l+\frac{r}{2}+\frac{1}{2}\right) & 0 & -\frac{3}{2}r
\end{pmatrix}. \tag{4.23}$$

Diagonalising now this matrix, we get the eigenvalues

$$\lambda_{[\frac{1}{2},\frac{1}{2}]} = \left\{ \left(\frac{1}{2} \pm \sqrt{H_0(r-1)+4}\right), \left(-\frac{1}{2} \pm \sqrt{H_0(r+1)+4}\right) \right\}, \tag{4.24}$$

where by $H_0(r\pm 1)$ we mean $H_0(j,l,r\pm 1)$. To (4.24) we have to add the four eigenvalues obtained from the inequivalent mass matrix, where one replaces r by $-r$.

The masses for the spinors and gravitinos are given in terms of $\not{D}$ by a numerical shift

$$\begin{aligned}
\text{gravitino :} \quad m(\psi_\mu) &= \not{D} - \frac{5}{2}; \\
\text{dilatino :} \quad m(\lambda) &= \not{D} + 1; \\
\text{longitudinal spinors:} \quad m(\psi^{(L)}) &= \not{D} + 3 .
\end{aligned} \tag{4.25}$$

This part of the spectrum has been first calculated in [5] and the values for the masses of these states agree with the ones we have written above.

The vector harmonic. The Laplace-Beltrami operator on the vector harmonics (3.2b) is

$$\boxtimes Y_a = \Box Y_a + 2R_a{}^b Y_b \,. \tag{4.26}$$

As can be easily seen from (3.21) to decompose a vector index under $\mathrm{U}_H(1)$, it is convenient to go to a complex basis, defining $(\pm) = 1 \pm i2$ and $(\hat{\pm}) = 3 \pm i4$.

In this new basis, the operator (4.26) becomes the matrix ($e = 1$)

$$\begin{pmatrix} \Box + \frac{21}{4} + \frac{3}{2} iT_5 & & & & \sqrt{6} iT_+ \\ & \Box + \frac{21}{4} - \frac{3}{2} iT_5 & & & -\sqrt{6} iT_- \\ & & \Box + \frac{21}{4} - \frac{3}{2} iT_5 & & -\sqrt{6} i\hat{T}_+ \\ & & & \Box + \frac{21}{4} + \frac{3}{2} iT_5 & \sqrt{6} i\hat{T}_- \\ \frac{\sqrt{6}}{2} iT_- & -\frac{\sqrt{6}}{2} iT_+ & -\frac{\sqrt{6}}{2} i\hat{T}_- & \frac{\sqrt{6}}{2} i\hat{T}_+ & \Box + 8 \end{pmatrix}$$

acting on the harmonics

$$Y_a^{\{\nu\}} = \begin{pmatrix} Y_{(+)} \\ Y_{(-)} \\ Y_{(\hat{+})} \\ Y_{(\hat{-})} \\ Y_{(0)} \end{pmatrix} = \begin{pmatrix} Y_{+1}^{(j,l,r+1)} \\ Y_{-1}^{(j,l,r-1)} \\ Y_{+1}^{(j,l,r-1)} \\ Y_{-1}^{(j,l,r+1)} \\ Y_0^{(j,l,r)} \end{pmatrix} . \tag{4.27}$$

Note that, in principle, the five entries of (4.27) are not independent because of the transversality condition $\mathcal{D}^a Y_a = 0$. However, it turns out to be more convenient to treat the five harmonics as independent, which amounts to say that we now consider also longitudinal harmonics of the form $\mathcal{D}_a Y$. The presence of a longitudinal harmonic means that among the five eigenvalues of the matrix, we should find the eigenvalue of the laplacean on the scalar harmonic, since the Laplace-Beltrami on a longitudinal p-form harmonic has the same eigenvalues of the Laplace-Beltrami operator acting on the $(p-1)$-form harmonic. In our case, we should then find the eigenvalue of the laplacean H_0 between the five we get. This indeed is the case and the remaining four eigenvalues for the (transverse) 1-forms are

$$\lambda_{[1,0]} = \{3 + H_0(j,l,r \pm 2), H_0 + 4 \pm 2\sqrt{H_0 + 4}\} \,. \tag{4.28}$$

The mass spectrum of the sixteen vectors is thus

$$m^2(a_\mu) = \begin{cases} 3 + H_0(j,l,r \pm 2)\,, \\ H_0 + 4 \pm 2\sqrt{H_0 + 4}\,, \end{cases} \tag{4.29}$$

$$m^2(B_\mu, \varphi_\mu) = \begin{cases} H_0(j,l,r \pm 2) + 7 \pm 4\sqrt{H_0 + 4}\,, \\ H_0 + 12 \pm 6\sqrt{H_0 + 4}\,, \\ H_0 + 4 \pm 2\sqrt{H_0 + 4}\,. \end{cases} \tag{4.30}$$

Actually, as the Laplace-Beltrami operator acts diagonally on the complex vector field $a_\mu(x)$, so we get eight mass values. Conversely, the vectors $B_\mu(x)$, $\varphi_\mu(x)$ get mixed in the linearised equations of motion [29], and upon diagonalisation we find two extra masses for each eigenvalue. As for the scalar fields, we will use the same names B_μ and ϕ_μ also for the eigenstates corresponding to linear combinations with plus or minus sign respectively in the mass formulae (4.30).

The two-form harmonic. The relevant Laplace-Beltrami operator is now of the first-order (3.3) and can be simply expressed as

$$\frac{1}{2}\epsilon^{abcde}\mathcal{D}_c Y_{de} = \frac{1}{2}\epsilon^{abcde}T_c Y_{de} + \epsilon^{abcde}(M_c)_d{}^s Y_{se}\,. \tag{4.31}$$

For the purpose of the computation it is useful to think the action of the generators in the representation space of a vector Y_{ab} labeled by a couple of antisymmetric indices.

Again, it is simpler to use a complex basis, where the various components of the tensor have a definite $\mathrm{U}_H(1)$ charge. Ordering the ten components of Y_{ab} as follows

$$\begin{pmatrix} Y_{+\hat{+}} \\ Y_{5+} \\ Y_{5\hat{+}} \\ Y_{+-} \\ Y_{\hat{+}\hat{-}} \\ Y_{+\hat{-}} \\ Y_{-\hat{+}} \\ Y_{5-} \\ Y_{5\hat{-}} \\ Y_{-\hat{-}} \end{pmatrix} = \begin{pmatrix} Y_{+2}^{(j,l,r)} \\ Y_{+1}^{(j,l,r+1)} \\ Y_{+1}^{(j,l,r-1)} \\ Y_{0}^{(j,l,r)} \\ Y_{0}^{(j,l,r)} \\ Y_{0}^{(j,l,r+2)} \\ Y_{0}^{(j,l,r-2)} \\ Y_{-1}^{(j,l,r-1)} \\ Y_{-1}^{(j,l,r+1)} \\ Y_{-2}^{(j,l,r)} \end{pmatrix} \tag{4.32}$$

and decomposing the free indices ab of (4.31) as in (4.32), the operator gives rise to the ten-dimensional vector

$$\begin{pmatrix} -2\sqrt{3}T_{\hat{+}}Y_{5+} + 2\sqrt{3}T_{+}Y_{5\hat{+}} + \frac{3}{\sqrt{2}}T_5 Y_{+\hat{+}} \\ 2i\sqrt{2}Y_{5+} + \sqrt{3}T_{\hat{+}}Y_{+\hat{-}} - \sqrt{3}T_{\hat{-}}Y_{+\hat{+}} - \sqrt{3}T_{+}Y_{\hat{+}\hat{-}} \\ -2i\sqrt{2}Y_{5\hat{+}} - \sqrt{3}T_{+}Y_{-\hat{+}} - \sqrt{3}T_{\hat{+}}Y_{+-} + \sqrt{3}T_{-}Y_{+\hat{+}} \\ 2\sqrt{3}T_{\hat{+}}Y_{5\hat{-}} - 2\sqrt{3}T_{\hat{-}}Y_{5\hat{+}} + \frac{3}{\sqrt{2}}T_5 Y_{\hat{+}\hat{-}} \\ 2\sqrt{3}T_{+}Y_{5-} - 2\sqrt{3}T_{-}Y_{5+} + \frac{3}{\sqrt{2}}T_5 Y_{+-} \\ -2\sqrt{3}T_{+}Y_{5\hat{-}} + 2\sqrt{3}T_{\hat{-}}Y_{5+} - \frac{3}{\sqrt{2}}T_5 Y_{+\hat{-}} + 3\sqrt{2}iY_{+\hat{-}} \\ 2\sqrt{3}T_{\hat{+}}Y_{5-} - 2\sqrt{3}T_{-}Y_{5\hat{+}} - \frac{3}{\sqrt{2}}T_5 Y_{-\hat{+}} - 3\sqrt{2}iY_{-\hat{+}} \\ -2i\sqrt{2}Y_{5-} - \sqrt{3}T_{\hat{+}}Y_{-\hat{-}} + \sqrt{3}T_{\hat{-}}Y_{-\hat{+}} + \sqrt{3}T_{-}Y_{\hat{+}\hat{-}} \\ 2i\sqrt{2}Y_{5\hat{-}} + \sqrt{3}T_{+}Y_{-\hat{-}} + \sqrt{3}T_{\hat{-}}Y_{+-} - \sqrt{3}T_{-}Y_{+\hat{-}} \\ -2\sqrt{3}T_{\hat{-}}Y_{5-} + 2\sqrt{3}T_{-}Y_{5\hat{-}} + \frac{3}{\sqrt{2}}T_5 Y_{-\hat{-}} \end{pmatrix}. \tag{4.33}$$

Comparing (4.33) with (4.32), one reconstructs the operator matrix $\boxtimes_{\xi\xi'}$ and then one computes the numerical matrix $M_{\xi\xi'}$. From this latter, we get four 0 eigenvalues corresponding to the longitudinal harmonics[5] and six eigenvalues corresponding to the transverse ones

$$\lambda_{[1,1]} = \left\{ i\left(1 \pm \sqrt{H_0(j,l,r\pm 2)+4}\right), \pm i\sqrt{H_0+4} \right\}. \tag{4.34}$$

The corresponding masses for the physical states are

$$m^2(b_{\mu\nu}) = \begin{cases} H_0 + 4\,, \\ H_0 + 4\,, \\ 5 + H_0(j,l,r\pm 2) \pm 2\sqrt{H_0(j,l,r\pm 2)+4}\,, \end{cases} \tag{4.35}$$

$$m^2(a) = \begin{cases} H_0 + 4 \pm 2\sqrt{H_0+4}\,, \\ H_0(j,l,r\pm 2) + 1 \pm 2\sqrt{H_0(j,l,r\pm 2)+4}\,. \end{cases} \tag{4.36}$$

Also in this case part of the spectrum we have shown has been computed in [5] and it agrees with our results.

The other harmonics. We have not calculated either the eigenvalues of $\not{D}$ corresponding to the vector-spinor harmonic Ξ_a which produce AdS_5 spinors $\psi^{(T)}$, or the eigenvalues of the symmetric traceless harmonic $Y^{(\nu)}_{(ab)}$. However, we know a priori how many states we obtain in these two cases, and by a counting argument we can circumvent the problem of the explicit computation of the eigenvalues of their mass matrices. For the vector-spinors we have in principle a matrix of rank 20, that becomes 16×16 due to the irreducibility condition, and further gets to 12×12, once the transversality condition $\mathcal{D}^a \Xi_a = 0$ is imposed. In this way we are left with 12 non-trivial (non longitudinal) eigenvalues and thus we expect 12 $\psi^{(T)}$ spinors. In an analogous way, the traceless symmetric tensor $Y^{(\nu)}_{(ab)}$ gives a 14×14 mass-matrix out of which five eigenvalues are longitudinal leaving 9 non-trivial eigenvalues.

If we match the bosonic and fermionic degrees of freedom including the $12 + 12$ (right) left-handed spinors $\psi^{(T)}$ and the 9 real fields ϕ of the traceless symmetric tensor we find 128 bosonic degrees of freedom and 128 fermionic ones. Therefore, once we have correctly and unambiguously assigned all the fields except the $\psi^{(T)}$ and ϕ to supermultiplets of $\mathrm{SU}(2,2|1)$, the remaining degrees of freedom of $\psi^{(T)}$ and ϕ are uniquely assigned to the supermultiplets for their completion.

5. Filling of $\mathrm{SU}(2,2|1)$ multiplets

This section aims at combining the results of the harmonic expansion on T^{11} with those of the purely group theoretical analysis of the $\mathrm{SU}(2,2|1)$ representations, and proceed by filling $\mathrm{SU}(2,2|1)$ supermultiplets with the appropriate eigenvalues of the

[5]Indeed in this case on a longitudinal harmonic $\mathcal{D}_{[a}Y_{b]}$ the operator is identically zero, i.e. $\epsilon^{abcde}\mathcal{D}_c\mathcal{D}_dY_e = 0$.

KK mass operators. This procedure was originally devised [31] in the analysis of the spectra of $\mathcal{N}=2$ supersymmetric AdS_4 compactifications of eleven dimensional supergravity, where the full symmetry group is OSp(4|2), and has recently been revisited in [11], leading to the uncovery of an interesting structure of new short multiplets. In the AdS_4 case, the reconstruction of the supermultiplets was advantaged by the results of [23] where universal mass relations among fields bound to each other by supersymmetry transformations were derived from the general properties of harmonic expansion on coset manifolds with Killing spinors [24]. For AdS_5 compactifications this tool is not available, but, as we will see, one can still fully assemble all the KK fields into multiplets and retrieve elsewhere all the necessary information.

On the group theory side, we need the unitary highest weight representations of SU(2, 2|1), originally worked out in [32], [33] and recently nicely popularized in an appendix of [34]. They are characterized in terms of the quantum numbers of the $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\subset\mathrm{SU}(2,2)\subset\mathrm{SU}(2,2|1)$ bosonic subalgebra (E_0, s_1, s_2), where E_0 is the AdS energy, and by the internal symmetry $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ labels (j, l, r), r being the R-symmetry charge.

The general relations between E_0 and the masses for fields of various spin are

$$\begin{aligned}
\text{spin 2:}\;\; E_0^{(2)} &= 2+\sqrt{4+m^2_{(2)}}\\
\text{spin }\frac{3}{2}:\; E_0^{(3/2)} &= 2+|m_{(3/2)}+3/2|\\
\text{spin 1:}\;\; E_0^{(1)} &= 2+\sqrt{1+m^2_{(1)}}\\
\text{two-form:}\;\; E_0^{(2f)} &= 2+|m_{(2f)}|\\
\text{spin }\frac{1}{2}:\; E_0^{(1/2)} &= 2\pm|m_{(1/2)}|\\
\text{spin 0:}\;\; E_0^{(0)} &= 2\pm\sqrt{4+m^2_{(0)}}
\end{aligned}\tag{5.1}$$

The sign ambiguity in the spin zero and 1/2 formulae occurs because in these specific cases the unitarity bound $E_0\geq 1+s$ allows the possibility $E_0<2$.

We will arrange our results in a serie of nine tables, summarizing the properties of the various families of unitary irreducible SU(2, 2|1) representations $\mathcal{D}(E_0, s_1, s_2; r)$, each generated by a specific (s_1, s_2) state, having $E_0^{(s)}=E_0$ and R-charge r. All descendant states have an $E_0^{(s)}$ value shifted in a range of ± 2 (in 1/2 steps) with respect to the E_0 of the multiplet, while their R-symmetry is shifted in a range of ± 2 in integer steps. The highest spin state has unshifted R-charge r. We have one graviton multiplet $\mathcal{D}(E_0, 1/2, 1/2; r)$ (table 2), two left $\mathcal{D}(E_0, 1/2, 0; r)$ and two right $\mathcal{D}(E_0, 0, 1/2; r)$ gravitino multiplets (tables 3–6) and four vector multiplets $\mathcal{D}(E_0, 0, 0; r)$ (tables 7–10). Thus their structure and the information in the columns labeled (s_1, s_2), $E_0^{(s)}$ and R-symm. is purely group theoretical and implied by the analysis of [32, 33, 34].

For generic values of the $SU(2) \times SU(2)$ quantum numbers j, l and the R-symmetry r, the multiplets of the tables 2–10 are massive long multiplets of $SU(2,2|1)$. However, it is well known [34] that multiplet shortening occurs for specific values of the $SU(2,2|1)$ quantum numbers, when $s_1 s_2 = 0$ or unitarity bounds are saturated. For $SU(2,2|1)$, there are three types of shortened representations:

$\diamond$ massless AdS multiplets,

$$E_0 = 2 + s_1 + s_2 \,, \qquad (s_1 - s_2) = \frac{3}{2} r \,, \tag{5.2}$$

$\star$ semi-long AdS multiplets:

$$E_0 = \begin{cases} \frac{3}{2} r + 2 s_2 + 2 \\ -\frac{3}{2} r + 2 s_1 + 2 \end{cases} \tag{5.3}$$

$\bullet$ chiral AdS multiplets

$$E_0 = \left| \frac{3}{2} r \right| . \tag{5.4}$$

These shortened representations are the most interesting ones in light of the correspondence with the CFT at the boundary, and their field theoretical counterparts have been extensively discussed in [1]. Shortening has been related to peculiar E_0 values reported in (5.2)–(5.4). The generic conditions to fulfill the requirements that leads to multiplet shortenings are thus

$$j = l = \left| \frac{r}{2} \right| , \qquad H_0 + 4 = \left(\frac{3}{2} r + 2 \right)^2 , \tag{5.5}$$

$$\left. \begin{array}{l} j = l - 1 = \left| \frac{r}{2} \right| \\ l = j - 1 = \left| \frac{r}{2} \right| \end{array} \right\} , \qquad H_0 + 4 = \left(\frac{3}{2} r + 4 \right)^2 , \tag{5.6}$$

to which it must be added the very peculiar one

$$j = l = \frac{r-2}{2} , \qquad r \geq 2 \qquad H_0 + 4 = \left(\frac{3}{2} r - 2 \right)^2 , \tag{5.7}$$

which gives rise to unitary shortenings only in one case [1].

We have thus added some symbols in the columns at the left of the tables to denote the surviving states in the shortened multiplets: chiral ($\bullet$), semi-long ($\star$) or massless ($\diamond$) multiplets.

JHEP11(1999)009

In particular, the absence of these symbols in table 4 means that no shortening of any kind can occur for the gravitino multiplet II.

As explained above, the columns regarding the (s_1, s_2), E_0, R-symmetry values and the surviving states under shortening are determined by purely group theoretical arguments. Our goal is therefore to fill in the remaining two right columns of all tables with the value of the masses (mass squared for all the fields related to second order operators), fitting unambiguously every KK field of the spectrum at the right place.

The method used to complete the multiplets is "by exhaustion" [11, 31]: it consists in starting from the highest spin 2 states, whose masses are unambiguously defined by the eigenvalue $\lambda_{[0,0]} = H_0$ and determining their energy label making use of (5.1). Once the E_0 of the multiplet (that in the graviton case belongs to one of the vector fields having R-charge r) is identified, one uses the inverses of (5.1) and 'predicts' the masses of all the remaining states, and identifies them among the KK mass eigenvalues. At the end of this process, after the whole graviton multiplet has been filled, part of the gravitino eigenvalues will still be unused. As they don't match any graviton, they must generate their own spin 3/2 multiplets, and one repeats the above procedure of arranging the lower spin states. Again, the leftover vectors eigenmodes will start vector multiplets and the filling process is reiterated until all the mass values have been used.

Note that in the graviton multiplet and the first two families of vector multiplets all masses are written in terms of the usual H_0

$$H_0(j,l,r) = 6\left(j(j+1) + l(l+1) - \frac{1}{8}r^2\right), \tag{5.8}$$

where r is the R charge of the fundamental state in the multiplet, while for the gravitino multiplets there appear the shifted quantities

$$H_0^{\pm} \equiv H_0(j,l,r\pm 1) = H_0 - \frac{3}{4}(1 \pm 2r) \tag{5.9}$$

and similar expressions

$$H_0^{\pm\pm} \equiv H_0(j,l,r\pm 2) = H_0 - 3(1 \pm r) \tag{5.10}$$

are used for the last two families of vector multiplets. This shifts are necessary in order to match the expected masses with formulae (4.11)–(4.25), obtained from the eigenvalues $\lambda_{[i,j]}$, where one happens to find H_0 with shifted r.

It is useful to list also the inverse of the formulae (5.1), that give directly the masses or mass squared of the various (s_1, s_2) representations, each having $(2s_1 +$

1)(2s_2 + 1) degrees of freedom:

$$
\begin{array}{rl}
(1,1) & m^2 = E_0(E_0-4)\,, \\
\left(\frac{1}{2},\frac{1}{2}\right) & m^2 = (E_0-1)(E_0-3)\,, \\
(1,0),(0,1) & m = |E_0-2|\,, \\
\left.\begin{array}{r} \left(\frac{1}{2},0\right),\left(0,\frac{1}{2}\right) \\ \left(\frac{1}{2},1\right),\left(1,\frac{1}{2}\right) \end{array}\right\} & m = |E_0-2|\,, \\
(0,0) & m^2 = E_0(E_0-4)\,,
\end{array}
\tag{5.11}
$$

For the sake of clarity, we give an example by explaining in detail the assembling of the graviton multiplet.

The first line in table 2 is filled without effort, since (4.11) tells that the graviton mass spectrum is given unambiguously by the scalar harmonic eigenvalue $m^2(H_{\mu\nu}) \equiv H_0(j,l,r)$, where r coincides with the R-charge of the basic vector state for the multiplet, having energy label $E_0^{(1)} = E_0$. Since $E_0^{(2)} = E_0+1$, we can derive by (5.1) that the energy label for the whole multiplet is

$$
E_0 = 1 + \sqrt{H_0+4}\,. \tag{5.12}
$$

Now we pass to the left and right gravitini, having generically mass spectrum given by (4.25)

$$
m(\psi_\mu^{L,R}) = \lambda_{[\frac{1}{2},\frac{1}{2}]} - \frac{5}{2} = \begin{cases} -2 \pm \sqrt{H_0(j,l,r\pm 1)+4} \\ -3 \pm \sqrt{H_0(j,l,r\pm 1)+4}\,. \end{cases} \tag{5.13}
$$

The predicted masses for each of the four spin 3/2 states are computed by inserting the appropriate energy, given in terms of (5.12), and R-charges in formulae (5.11). Due to the presence of absolute values (and below, for spinors and scalars also of double sign choices), the set found in this way is redundant. Among the eight choices of eigenmodes (5.13), only the four upper ones have a match within the predicted set of the graviton multiplet, while the remaining lower ones will be highest spin states for gravitino multiplets.

The same procedure is applied to identify the four vector states, by matching formulae (5.11) with the available spin one fields and eigenvalues within the choices (4.29)–(4.30), according to the different energy and R-charges. We point out that while the a_μ field is selected without ambiguity, we have chosen to assign the B_μ field with the positive signs in (4.30) and ϕ_μ the negative ones. The remaining eigenvalues will either fit in gravitino multiplets or give rise to vector multiplets on their own.

Analogously, the 2-form fields, spinors and the scalars are placed giving rise to the generic massive *long* graviton multiplet, existing for arbitrary quantum numbers (j,l,r) and having an irrational value of E_0.

Then we consider the various group-theoretical shortening patterns. If condition (5.5) is imposed, some of the states drop out of the graviton multiplet that reduces to the semi-long multiplet identified with the $\star$ symbols.

The massless graviton multiplet identified by the $\diamond$ is obtained by further setting $j = l = r = 0$, leading to $H_0 = 0$ and $E_0 = 3$.

The application of the above method yields the completion of all other tables, whose shortening patterns have been treated to great depth in [1], to which we refer for any detail.

		(s_1, s_2)	$E_0^{(s)}$	R-symm.	field	Mass
$\diamond$	$\star$	(1,1)	E_0+1	r	$g_{\mu\nu}$	H_0
$\diamond$	$\star$	(1,1/2)	$E_0+1/2$	$r-1$	ψ_μ^L	$-2+\sqrt{H_0+4}$
$\diamond$	$\star$	(1/2,1)	$E_0+1/2$	$r+1$	ψ_μ^R	$-2+\sqrt{H_0+4}$
	$\star$	(1/2,1)	$E_0+3/2$	$r-1$	ψ_μ^R	$-2-\sqrt{H_0+4}$
		(1,1/2)	$E_0+3/2$	$r+1$	ψ_μ^L	$-2-\sqrt{H_0+4}$
$\diamond$	$\star$	(1/2,1/2)	E_0	r	ϕ_μ	$H_0+4-2\sqrt{H_0+4}$
		(1/2,1/2)	E_0+1	$r+2$	a_μ	H_0+3
	$\star$	(1/2,1/2)	E_0+1	$r-2$	a_μ	H_0+3
		(1/2,1/2)	E_0+2	r	B_μ	$H_0+4+2\sqrt{H_0+4}$
		(1,0)	E_0+1	r	$b_{\mu\nu}^+$	$\sqrt{H_0+4}$
	$\star$	(0,1)	E_0+1	r	$b_{\mu\nu}^-$	$-\sqrt{H_0+4}$
		(1/2,0)	$E_0+1/2$	$r+1$	λ_L	$1/2-\sqrt{H_0+4}$
	$\star$	(0,1/2)	$E_0+1/2$	$r-1$	λ_R	$1/2-\sqrt{H_0+4}$
		(1/2,0)	$E_0+3/2$	$r-1$	λ_L	$1/2+\sqrt{H_0+4}$
		(0,1/2)	$E_0+3/2$	$r+1$	λ_R	$1/2+\sqrt{H_0+4}$
		(0,0)	E_0+1	r	B	H_0

Table 2: Graviton multiplet, $E_0 = 1+\sqrt{H_0+4}$.

		(s_1, s_2)	$E_0^{(s)}$	R-symm.	field	Mass
	$\star$	(1,1/2)	$E_0 + 1$	r	ψ_μ^L	$-3 + \sqrt{H_0^- + 4}$
	$\star$	(1/2,1/2)	$E_0 + 1/2$	$r + 1$	ϕ_μ	$H_0^- + 7 - 4\sqrt{H_0^- + 4}$
	$\star$	(1/2,1/2)	$E_0 + 3/2$	$r - 1$	a_μ	$H_0^- + 4 - 2\sqrt{H_0^- + 4}$
$\bullet$	$\star$	(1,0)	$E_0 + 1/2$	$r - 1$	$a_{\mu\nu}$	$2 - \sqrt{H_0^- + 4}$
		(1,0)	$E_0 + 3/2$	$r + 1$	$b_{\mu\nu}^+$	$1 - \sqrt{H_0^- + 4}$
$\bullet$	$\star$	(1/2,0)	E_0	r	$\psi_L^{(T)}$	$-5/2 + \sqrt{H_0^- + 4}$
$\bullet$	$\star$	(1/2,0)	$E_0 + 1$	$r - 2$	$\psi_L^{(T)}$	$-3/2 + \sqrt{H_0^- + 4}$
	$\star$	(0,1/2)	$E_0 + 1$	r	λ_R	$3/2 - \sqrt{H_0^- + 4}$
		(1/2,0)	$E_0 + 1$	$r + 2$	$\psi_L^{(T)}$	$-3/2 + \sqrt{H_0^- + 4}$
		(1/2,0)	$E_0 + 2$	r	$\psi_L^{(T)}$	$-1/2 + \sqrt{H_0^- + 4}$
$\bullet$	$\star$	(0,0)	$E_0 + 1/2$	$r - 1$	a	$H_0^- + 4 - 4\sqrt{H_0^- + 4}$
		(0,0)	$E_0 + 3/2$	$r + 1$	a	$H_0^- + 1 - 2\sqrt{H_0^- + 4}$

Table 3: Gravitino multiplet I, $E_0 = \sqrt{H_0^- + 4} - 1/2$.

(s_1, s_2)	$E_0^{(s)}$	R-symm.	field	Mass
(1,1/2)	$E_0 + 1$	r	ψ_μ^L	$-3 - \sqrt{H_0^+ + 4}$
(1/2,1/2)	$E_0 + 1/2$	$r + 1$	a_μ	$H_0^+ + 4 + 2\sqrt{H_0^+ + 4}$
(1/2,1/2)	$E_0 + 3/2$	$r - 1$	B_μ	$H_0^+ + 7 + 4\sqrt{H_0^+ + 4}$
(1,0)	$E_0 + 1/2$	$r - 1$	$b_{\mu\nu}^+$	$1 + \sqrt{H_0^+ + 4}$
(1,0)	$E_0 + 3/2$	$r + 1$	$a_{\mu\nu}$	$2 + \sqrt{H_0^+ + 4}$
(1/2,0)	E_0	r	$\psi_L^{(T)}$	$-1/2 - \sqrt{H_0^+ + 4}$
(1/2,0)	$E_0 + 1$	$r - 2$	$\psi_L^{(T)}$	$-3/2 - \sqrt{H_0^+ + 4}$
(0,1/2)	$E_0 + 1$	r	λ_R	$3/2 + \sqrt{H_0^+ + 4}$
(1/2,0)	$E_0 + 1$	$r + 2$	$\psi_L^{(T)}$	$-3/2 - \sqrt{H_0^+ + 4}$
(1/2,0)	$E_0 + 2$	r	$\psi_L^{(T)}$	$-5/2 - \sqrt{H_0^+ + 4}$
(0,0)	$E_0 + 1/2$	$r - 1$	a	$H_0^+ + 1 + 2\sqrt{H_0^+ + 4}$
(0,0)	$E_0 + 3/2$	$r + 1$	a	$H_0^+ + 4 + 4\sqrt{H_0^+ + 4}$

Table 4: Gravitino multiplet II, $E_0 = 5/2 + \sqrt{H_0^+ + 4}$.

	(s_1,s_2)	$E_0^{(s)}$	R-symm.	field	Mass
$\star$	(1/2,1)	E_0+1	r	ψ_μ^R	$-3+\sqrt{H_0^+ +4}$
$\star$	(1/2,1/2)	$E_0+1/2$	$r-1$	ϕ_μ	$H_0^+ +7-4\sqrt{H_0^+ +4}$
	(1/2,1/2)	$E_0+3/2$	$r+1$	a_μ	$H_0^+ +4-2\sqrt{H_0^+ +4}$
$\star$	(0,1)	$E_0+1/2$	$r+1$	$a_{\mu\nu}$	$2-\sqrt{H_0^+ +4}$
$\star$	(0,1)	$E_0+3/2$	$r-1$	$b^-_{\mu\nu}$	$1-\sqrt{H_0^+ +4}$
$\star$	(0,1/2)	E_0	r	$\psi_R^{(T)}$	$-5/2+\sqrt{H_0^+ +4}$
$\star$	(0,1/2)	E_0+1	$r+2$	$\psi_R^{(T)}$	$-3/2+\sqrt{H_0^+ +4}$
	(1/2,0)	E_0+1	r	λ_L	$3/2-\sqrt{H_0^+ +4}$
	(0,1/2)	E_0+1	$r-2$	$\psi_R^{(T)}$	$-3/2+\sqrt{H_0^+ +4}$
	(0,1/2)	E_0+2	r	$\psi_R^{(T)}$	$-1/2+\sqrt{H_0^+ +4}$
	(0,0)	$E_0+1/2$	$r+1$	a	$H_0^+ +4-4\sqrt{H_0^+ +4}$
	(0,0)	$E_0+3/2$	$r-1$	a	$H_0^+ +1-2\sqrt{H_0^+ +4}$

Table 5: Gravitino multiplet III, $E_0=-1/2+\sqrt{H_0^+ +4}$.

	(s_1,s_2)	$E_0^{(s)}$	R-symm.	field	Mass
$\star$	(1/2,1)	E_0+1	r	ψ_μ^R	$-3-\sqrt{H_0^- +4}$
$\star$	(1/2,1/2)	$E_0+1/2$	$r-1$	a_μ	$H_0^- +4+2\sqrt{H_0^- +4}$
	(1/2,1/2)	$E_0+3/2$	$r+1$	B_μ	$H_0^- +7+4\sqrt{H_0^- +4}$
$\star$	(0,1)	$E_0+1/2$	$r+1$	$b^-_{\mu\nu}$	$1+\sqrt{H_0^- +4}$
$\star$	(0,1)	$E_0+3/2$	$r-1$	$a_{\mu\nu}$	$2+\sqrt{H_0^- +4}$
$\star$	(0,1/2)	E_0	r	$\psi_R^{(T)}$	$-1/2-\sqrt{H_0^- +4}$
$\star$	(0,1/2)	E_0+1	$r+2$	$\psi_R^{(T)}$	$-3/2-\sqrt{H_0^- +4}$
	(1/2,0)	E_0+1	r	λ_L	$3/2+\sqrt{H_0^- +4}$
	(0,1/2)	E_0+1	$r-2$	$\psi_R^{(T)}$	$-3/2-\sqrt{H_0^- +4}$
	(0,1/2)	E_0+2	r	$\psi_R^{(T)}$	$-5/2-\sqrt{H_0^- +4}$
	(0,0)	$E_0+1/2$	$r+1$	a	$H_0^- +1+2\sqrt{H_0^- +4}$
	(0,0)	$E_0+3/2$	$r-1$	a	$H_0^- +4+4\sqrt{H_0^- +4}$

Table 6: Gravitino multiplet IV, $E_0=5/2+\sqrt{H_0^- +4}$.

			(s_1,s_2)	$E_0^{(s)}$	R-symm.	field	Mass
$\diamond$		$\star$	(1/2,1/2)	E_0+1	r	ϕ_μ	$H_0+12-6\sqrt{H_0+4}$
$\diamond$	$\bullet$	$\star$	(1/2,0)	$E_0+1/2$	$r-1$	$\psi_L^{(L)}$	$7/2-\sqrt{H_0+4}$
$\diamond$		$\star$	(0,1/2)	$E_0+1/2$	$r+1$	$\psi_R^{(L)}$	$7/2-\sqrt{H_0+4}$
		$\star$	(0,1/2)	$E_0+3/2$	$r-1$	$\psi_R^{(L)}$	$5/2-\sqrt{H_0+4}$
			(1/2,0)	$E_0+3/2$	$r+1$	$\psi_L^{(L)}$	$5/2-\sqrt{H_0+4}$
$\diamond$	$\bullet$	$\star$	(0,0)	E_0	r	b	$H_0+16-8\sqrt{H_0+4}$
	$\bullet$	$\star$	(0,0)	E_0+1	$r-2$	ϕ	$H_0+9-6\sqrt{H_0+4}$
			(0,0)	E_0+1	$r+2$	ϕ	$H_0+9-6\sqrt{H_0+4}$
			(0,0)	E_0+2	r	ϕ	$H_0+4-4\sqrt{H_0+4}$

Table 7: Vector multiplet I, $E_0=\sqrt{H_0+4}-2$.

(s_1,s_2)	$E_0^{(s)}$	R-symm.	field	Mass
(1/2,1/2)	E_0+1	r	B_μ	$H_0+12+6\sqrt{H_0+4}$
(1/2,0)	$E_0+1/2$	$r-1$	$\psi_L^{(L)}$	$5/2+\sqrt{H_0+4}$
(0,1/2)	$E_0+1/2$	$r+1$	$\psi_R^{(L)}$	$5/2+\sqrt{H_0+4}$
(0,1/2)	$E_0+3/2$	$r-1$	$\psi_R^{(L)}$	$7/2+\sqrt{H_0+4}$
(1/2,0)	$E_0+3/2$	$r+1$	$\psi_L^{(L)}$	$7/2+\sqrt{H_0+4}$
(0,0)	E_0	r	ϕ	$H_0+4+4\sqrt{H_0+4}$
(0,0)	E_0+1	$r-2$	ϕ	$H_0+9+6\sqrt{H_0+4}$
(0,0)	E_0+1	$r+2$	ϕ	$H_0+9+6\sqrt{H_0+4}$
(0,0)	E_0+2	r	π	$H_0+16+8\sqrt{H_0+4}$

Table 8: Vector multiplet II, $E_0=\sqrt{H_0+4}+4$.

	(s_1,s_2)	$E_0^{(s)}$	R-symm.	field	Mass
	(1/2,1/2)	E_0+1	r	a_μ	$H_0^{++}+3$
	(1/2,0)	$E_0+1/2$	$r-1$	$\psi_L^{(T)}$	$+1/2-\sqrt{H_0^{++}+4}$
	(0,1/2)	$E_0+1/2$	$r+1$	$\psi_R^{(T)}$	$-1/2+\sqrt{H_0^{++}+4}$
	(0,1/2)	$E_0+3/2$	$r-1$	$\psi_R^{(T)}$	$1/2+\sqrt{H_0^{++}+4}$
•	(1/2,0)	$E_0+3/2$	$r+1$	$\psi_L^{(T)}$	$-1/2-\sqrt{H_0^{++}+4}$
	(0,0)	E_0	r	a	$H_0^{++}+1-2\sqrt{H_0^{++}+4}$
	(0,0)	E_0+1	$r-2$	ϕ	H_0^{++}
•	(0,0)	E_0+1	$r+2$	ϕ	H_0^{++}
•	(0,0)	E_0+2	r	a	$H_0^{++}+1+2\sqrt{H_0^{++}+4}$

Table 9: Vector multiplet III, $E_0=\sqrt{H_0^{++}+4}+1$.

		(s_1,s_2)	$E_0^{(s)}$	R-symm.	field	Mass
	$\star$	(1/2,1/2)	E_0+1	r	a_μ	$H_0^{--}+3$
•	$\star$	(1/2,0)	$E_0+1/2$	$r-1$	$\psi_L^{(T)}$	$-1/2+\sqrt{H_0^{--}+4}$
	$\star$	(0,1/2)	$E_0+1/2$	$r+1$	$\psi_R^{(T)}$	$+1/2+\sqrt{H_0^{--}+4}$
	$\star$	(0,1/2)	$E_0+3/2$	$r-1$	$\psi_R^{(T)}$	$-1/2-\sqrt{H_0^{--}+4}$
		(1/2,0)	$E_0+3/2$	$r+1$	$\psi_L^{(T)}$	$1/2+\sqrt{H_0^{--}+4}$
•	$\star$	(0,0)	E_0	r	a	$H_0^{--}+1-2\sqrt{H_0^{--}+4}$
•	$\star$	(0,0)	E_0+1	$r-2$	B	H_0^{--}
		(0,0)	E_0+1	$r+2$	ϕ	H_0^{--}
		(0,0)	E_0+2	r	a	$H_0^{--}+1+2\sqrt{H_0^{--}+4}$

Table 10: Vector multiplet IV, $E_0=\sqrt{H_0^{--}+4}+1$.

6. The Betti 3-form

In this last section we want to add some considerations on the existence of the Betti vector multiplet in the KK spectrum previously discussed. It is obvious that Betti multiplets always exist when we expand in harmonics a $(p+1)$-form on the higher-dimensional space considered as a vector on AdS_{D-d} and a p-form on the internal compact space X_d provided X_d contains non-trivial homology p-cycles (or $(D-p)$-cycles by Poincaré duality). Indeed, the first appearance was discussed in KK $AdS_4 \times S^7$ compactification [23], while the corresponding physical meaning was first discussed in [35] where the Betti vector multiplets have been interpreted as topological modes corresponding to the wrapping of p-branes around the internal p-cycles.

The interesting fact emerging from the analysis we presented is that in the AdS_5 case we have not only the single Betti vector multiplet but also a Betti tensor and a Betti hypermultiplet deriving from the $A_{\mu\nu ab}$ and A_{ab} fields, which are expanded in terms of the 2-form harmonic $Y_{[ab]}$ containing the 2-form dual to the Betti form.

In this section we determine the explicit form of the Betti 3-form on T^{11} using the theorems proven in [23, 30]. They are

1. The Betti p-form is valued in the holonomy algebra of the internal space X_d;

2. The Betti p-form is in the singlet representation of the isometry group G.

Let us first determine the holonomy algebra on T^{11}.

6.1 The holonomy algebra

We have already shown with the harmonic analysis the existence of two Killing spinors for the T^{11} space, implying $\mathcal{N}=2$ supersymmetry in the bulk. We want now to refine this result by explicitly deriving the holonomy algebra $\mathcal{H}$.

A Killing spinor is a zero mode of all the holonomy algebra generators

$$T_{\mathcal{H}}\eta = 0 \tag{6.1}$$

and therefore the number of preserved supersymmetries is equal to the number of independent solutions of the above equation.

The holonomy algebra must be a subalgebra of the tangent space one $\mathcal{H} \subset$ SO(5) and in order to have $\mathcal{N}=2$ supersymmetry $\mathcal{H}$ must be equal to SU(2) [26], corresponding to the canonical embedding SO(3) $\hookrightarrow$ SO(5):

$$\underline{5} \to \underline{3} + \underline{1} + \underline{1}\,. \tag{6.2}$$

Let us derive this result by explicitly building the $T_{\mathcal{H}}$ generators annihilating the Killing spinors. The $T_{\mathcal{H}}$ generators can be constructed by the analysis of the integrability condition of the Killing spinor equation

$$\mathcal{D}_a \eta = \frac{e}{2} \gamma_a \eta \,. \tag{6.3}$$

This equation admits solutions if and only if it is integrable, i.e.

$$[\mathcal{D}_a, \mathcal{D}_b]\eta = \frac{1}{4} R_{ab}{}^{cd} \gamma_{cd} \eta = \frac{e^2}{4} \gamma_{ab} \eta \,, \tag{6.4}$$

which can be recast in the simpler expression

$$(R_{ab,cd} \gamma^{cd} - \gamma_{ab}) \eta = 0 \,. \tag{6.5}$$

It is now straightforward to implement this equation by inserting the Riemann curvature definitions (2.16). Decomposing the antisymmetric couple ab according to $a, b = i, s, 5$, we find

$$\begin{aligned} ij: &\quad (3\gamma_{12} + 2\gamma_{34} - \gamma_{12})\eta = 0 \,, \\ st: &\quad (3\gamma_{34} + 2\gamma_{12} - \gamma_{34})\eta = 0 \,, \\ i5: &\quad 0 = 0 \,, \\ s5: &\quad 0 = 0 \,, \\ is: &\quad (\gamma_{24} - \gamma_{13})\eta = 0 \,, \\ &\quad (\gamma_{23} + \gamma_{14})\eta = 0 \,. \end{aligned} \tag{6.6}$$

The three independent elements of $T_{\mathcal{H}}$ (fixing the normalisation in a convenient way) are thus given by

$$\begin{aligned} g_1 &= -\frac{1}{2}(\gamma_{12} + \gamma_{34}) \,, \\ g_2 &= -\frac{1}{2}(\gamma_{24} - \gamma_{13}) \,, \\ g_3 &= -\frac{1}{2}(\gamma_{23} + \gamma_{14}) \,, \end{aligned} \tag{6.7}$$

and using the gamma matrix algebra, one easily concludes that the g_i close the SU(2) algebra

$$[g_i, g_j] = \epsilon_{ijk} g_k \,. \tag{6.8}$$

Recalling that the Killing spinors (4.17) derived previously from group theoretical arguments are singlet under the holonomy algebra, setting

$$h = \gamma^{abc} \Omega_{abc} \,, \tag{6.9}$$

then h annihilates all the Killing spinors: $h\eta = 0$.

6.2 The Betti 3-form

It is known [30] that a harmonic 3-form Ω on the T^{11} manifold must be in the singlet representation of the isometry group. This implies that its quantum numbers are $j = l = r = 0$.

Using the duality relation on the forms, the element (6.9) becomes

$$\tilde{h} = \gamma^{ab}\tilde{\Omega}_{ab}\,. \tag{6.10}$$

Recalling the analysis of the $\mathrm{U}_H(1)$ decomposition of the 2-form given in section 4, eq. (4.32), and taking into account that h must be a singlet of the full isometry group $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}_R(1)$, the only components of Ω_{ab} differing from zero are $\tilde{\Omega}_{+-}$ and $\tilde{\Omega}_{\hat{+}\hat{-}}$ (in complex notation) or $\tilde{\Omega}_{12}$ and $\tilde{\Omega}_{34}$ (in real notation) and that they must take real constant values (Since they contain only the singlet harmonic $Y^0_{(0)}$ in their expansion).

If we set

$$\tilde{\Omega}_{+-} = a\,, \qquad \tilde{\Omega}_{\hat{+}\hat{-}} = b\,, \qquad a, b \in \mathbb{R}\,, \tag{6.11}$$

then the condition for h to be valued in the holonomy algebra $\mathcal{H}$ reads

$$\tilde{h} = a\gamma^{12} + b\gamma^{34} = \alpha g_1 + \beta g_2 + \gamma g_3\,, \tag{6.12}$$

whose solution is $\alpha = a = b$ and $\beta = \gamma = 0$ implying that $\tilde{h} = a\,g_1$. Thus, fixing $a = 1$ for convenience, the Betti form is simply

$$\Omega = \star\left[V^1V^2 + V^3V^4\right]. \tag{6.13}$$

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A. Notations and conventions

Consider $AdS_5\times T^{11}$. We call M, N the curved ten-dimensional indices, $\mu,\nu/m,n$ the curved/flat AdS_5 ones and $\alpha\beta/a,b$ the curved/flat T^{11} ones.

Our ten-dimensional metric is the mostly minus $\eta = \{+-\cdots-\}$, so that the internal space has a negative definite metric. For ease of construction, we have also used a negative metric to raise and lower the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ Lie-algebra indices.

Furthermore, for the SU(2) algebras we defined $\epsilon^{123} = \epsilon^{12} = 1$.
The SO(5) gamma matrices are

$$\gamma_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}, \tag{A.1}$$

$$\gamma_3 = \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} & & i & \\ & & & i \\ i & & & \\ & i & & \end{pmatrix}, \tag{A.2}$$

$$\gamma_5 = \begin{pmatrix} i & & & \\ & i & & \\ & & -i & \\ & & & -i \end{pmatrix}. \tag{A.3}$$

B. The algebraic reduction of the covariant derivative

As in section 3.1, given a representation $\{\nu\}$ of G, we call the index spanning the representation space m ($m = 1, \ldots, \dim\{\nu\}$), while we denote by h_i an index ranging in the subset spanned by the fragment $\{\alpha_i\}$ ($h_i = 1, \ldots, \dim\{\alpha_i\}$).

Let us start from the left-invariant one-form on G/H decomposed along the $T_H \in \mathbb{H}$ and $T_a \in \mathbb{K}$ generators according to the Lie algebra decomposition $\mathbb{G} = \mathbb{H} + \mathbb{K}$:

$$L^{-1}dL = \omega^H T_H + V^a T_a \,, \tag{B.1}$$

where V^a are the vielbeins of G/H and ω^H is the so-called H-connection. We have

$$\mathcal{D}L^{-1} = -V^a(T_a L^{-1} + \omega_a^H T_H L^{-1}) \,. \tag{B.2}$$

We have by definition of harmonic $(L^{-1})^m{}_{h_i} \equiv Y^{\{\nu\}m}_{h_i}$. From (B.2) we get

$$\begin{aligned} (T_a L^{-1})^m{}_{h_i} &= (T_a)^n{}_{h_i}(L^{-1})^m{}_n = \sum_{j=1}^{M} (T_a)_{h_i}{}^{h_j} Y^{\{\nu\}m}_{h_j} \\ (T_H L^{-1})^m{}_{h_i} &= (T_H)^{k_i}{}_{h_i}(L^{-1})^m{}_{k_i} = (T_H)_{h_i}{}^{k_i} Y^{\{\nu\}m}_{k_i} \,, \end{aligned} \tag{B.3}$$

where in the latter we have used the fact that, being $\{\alpha_i\}$ irreducible, only the entries in the i-th block $(T_h)^{k_i}_{h_i}$ are non-vanishing.

Hence, (B.2) becomes (omitting the index $\{\nu\}$ denoting the G-representation)

$$dY^m_{h_i} = -V^a \left(\sum_{j=1}^{N} (T_a)_{h_i}{}^{h_j} Y^m_{h_i} + \omega_a^H (T_H)_{h_i}{}^{k_i} Y^m_{h_i} \right) \tag{B.4}$$

or, introducing the H-covariant derivative

$$\mathcal{D}^H \equiv \mathcal{D}(\omega^H) \tag{B.5}$$

we have

$$\mathcal{D}^H Y^m_{h_i} = -r(a) V^a \sum_{j=1}^{N} (T_a)_{h_i}{}^{h_j} Y^m_{h_i} \,, \tag{B.6}$$

where we have taken into account the vielbein rescaling $r(a)$ introduced in (2.11). this is formula (3.31) of the text.

On the other hand, an SO(d) harmonic (see equation (3.25)) has SO(d) covariant derivative given by

$$\mathcal{D}Y^m_{h_i} = dY^m_{h_i} + (\mathcal{B}^{cd})(T_{cd})_{h_i}{}^{k_i} Y^m_{k_i} \,. \tag{B.7}$$

where again we have used the fact that T_{ab} is block-diagonal under the branching SO(d) $\to H$. Decomposing $\mathcal{B}^{ab}$ into the H-connection plus more we have

$$\mathcal{B}^{cd} T_{cd} = \omega^H T_H + M^{cd} T_{cd} \tag{B.8}$$

and substituting in (B.7) we obtain

$$\mathcal{D}Y^m_{h_i} = r(a) V^a \sum_{j=1}^{N} (T_a)_{h_i}{}^{h_j} Y^m_{h_j} + M^{cd} (T_{cd})_{h_i}{}^{k_i} Y^m_{k_i} \,, \tag{B.9}$$

which gives rise to equation (3.32) of the text.

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